Simultaneous Feedback Edge Set: A Parameterized Perspective

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In a recent article Agrawal et al. (STACS 2016) studied a simultaneous variant of the classic FEEDBACK VERTEX SET problem, called SIMULTANEOUS FEEDBACK VERTEX SET (SIM-FVS). In this problem the input is an n-vertex graph G, an integer k and a coloring function $\mathsf{col} : E(G) \rightarrow 2^{[\alpha]}$, and the objective is to check whether there exists a vertex subset S of cardinality at most k in G such that for all $i \in [\alpha]$, $G_i - S$ is acyclic. Here, $G_i = (V(G), \{e \in E(G) \mid i \in \mathsf{col}(e)\})$ and $[\alpha] = \{1, \ldots, \alpha\}$. In this paper we consider the edge variant of the problem, namely, SIMULTANEOUS FEEDBACK EDGE SET (SIM-FES). In this problem, the input is same as the input of SIM-FVS and the objective is to check whether there is an edge subset S of cardinality at most k in G such that for all $i \in [\alpha]$, $G_i - S$ is acyclic. Unlike the vertex variant of the problem, when $\alpha = 1$, the problem is equivalent to finding a maximal spanning forest and hence it is polynomial time solvable. We show that for $\alpha = 3$ SIM-FES is NP-hard by giving a reduction from VERTEX COVER on cubic-graphs. The same reduction shows that the problem does not admit an algorithm of running time $\mathcal{O}(2^{o(k)} n^{\mathcal{O}(1)})$ unless ETH fails. This hardness result is complimented by an FPT algorithm for SIM-FES running in time $\mathcal{O}(2^{\omega k\alpha + \alpha \log k} n^{\mathcal{O}(1)})$, where ω is the exponent in the running time of matrix multiplication. The same algorithm gives a polynomial time algorithm for the case when $\alpha = 2$. We also give a kernel for SIM-FES with $(k\alpha)^{\mathcal{O}(\alpha)}$ vertices. Finally, we consider the problem MAXIMUM SIMULTANEOUS ACYCLIC SUBGRAPH. Here, the input is a graph G, an integer q and, a coloring function $\mathsf{col} : E(G) \rightarrow 2^{[\alpha]}$. The question is whether there is an edge subset F of cardinality at least q in G such that for all $i \in [\alpha]$, $G[F_i]$ is acyclic. Here, $F_i = \{e \in F \mid i \in \mathsf{col}(e)\}$. We give an FPT algorithm for MAXIMUM SIMULTANEOUS ACYCLIC SUBGRAPH running in time $\mathcal{O}(2^{\omega q\alpha} n^{\mathcal{O}(1)})$. All our algorithms are based on a parameterized version of the MATROID PARITY problem.

CCS Concepts: •**Mathematics of computing** → **Graph algorithms;** Matroids and greedoids; •**Theory of computation** → **Fixed parameter tractability;**

Additional Key Words and Phrases: Parameterized Complexity, Feedback edge set, α-matroid parity

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1. INTRODUCTION

Deleting at most k vertices or edges from a given graph G, so that the resulting graph belongs to a particular family of graphs ($\mathcal{F}$), is an important research direction in the fields of graph algorithms and parameterized complexity. For a family of graphs

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$\mathcal{F}$, given a graph G and an integer k, the $\mathcal{F}$-DELETION (EDGE $\mathcal{F}$-DELETION) problem asks whether we can delete at most k vertices (edges) in G so that the resulting graph belongs to $\mathcal{F}$. The $\mathcal{F}$-DELETION (EDGE $\mathcal{F}$-DELETION) problems generalize many of the NP-hard problems like VERTEX COVER, FEEDBACK VERTEX SET, ODD CYCLE TRANSVERSAL, EDGE BIPARTIZATION, etc. Inspired by applications, Cai and Ye introduced variants of $\mathcal{F}$-DELETION (EDGE $\mathcal{F}$-DELETION) on edge colored graphs [Cai and Ye 2014]. One of the natural generalizations to the classic $\mathcal{F}$-DELETION (EDGE $\mathcal{F}$-DELETION) problems on edge colored graphs is the following. Given a graph G with a coloring function $\mathsf{col} : E(G) \to 2^{[\alpha]}$, and an integer k, we want to delete a set S of at most k edges/vertices in G so that for each $i \in [\alpha]$, $G_i - S$ belongs to $\mathcal{F}$. Here, G_i is the graph with vertex set $V(G)$ and edge set as $\{e \in E(G) \mid i \in \mathsf{col}(e)\}$. These problems are also called *simultaneous* variant of $\mathcal{F}$-DELETION (EDGE $\mathcal{F}$-DELETION).

Cai and Ye studied the DUALLY CONNECTED INDUCED SUBGRAPH and DUAL SEPARATOR on 2-edge colored graphs [Cai and Ye 2014]. Agrawal et al. [Agrawal et al. 2016] studied a simultaneous variant of FEEDBACK VERTEX SET, called SIMULTANEOUS FEEDBACK VERTEX SET, in the realm of parameterized complexity. Here, the input is a graph G, an integer k, and a coloring function $\mathsf{col} : E(G) \to 2^{[\alpha]}$ and the objective is to check whether there is a set S of at most k vertices in G such that for all $i \in [\alpha]$, $G_i - S$ is acyclic. Here, $G_i = (V(G), \{e \in E(G) \mid i \in \mathsf{col}(e)\})$. In this paper we consider the edge variant of the problem, namely, SIMULTANEOUS FEEDBACK EDGE SET, in the realm of parameterized complexity.

In the Parameterized Complexity paradigm the main objective is to design an algorithm with running time $f(\mu) \cdot n^{\mathcal{O}(1)}$, where μ is the parameter associated with the input, n is the size of the input and $f(\cdot)$ is some computable function whose value depends only on μ. A problem which admits such an algorithm is said to be *fixed parameter tractable* parameterized by μ. Typically, for edge/vertex deletion problems one of the natural parameters that is associated with the input is the size of the solution we are looking for. Another objective in parameterized complexity is to design polynomial time pre-processing routines that reduce the size of the input as much as possible. The notion of such a pre-processing routine is captured by *kernelization* algorithms. A kernelization algorithm for a parameterized problem Q takes as input an instance (I, k) of Q, runs in polynomial time and returns an equivalent instance (I', k') of Q. Moreover, the size of the instance (I', k') returned by the kernelization algorithm is bounded by $g(k)$, where $g(\cdot)$ is some computable function whose value depends only on k. If $g(\cdot)$ is polynomial in k, then the problem Q is said to admit a polynomial kernel. The instance returned by the kernelization is referred to as a *kernel* or a reduced instance. We refer the readers to the recent book of Cygan et al. [Cygan et al. 2015] for a more detailed overview of parameterized complexity and kernelization.

A *feedback edge set* in a graph G is $S \subseteq E(G)$ such that $G - S$ is a forest. For a graph G with a coloring function $\mathsf{col} : E(G) \to 2^{[\alpha]}$, a *simultaneous feedback edge set* is a subset $S \subseteq E(G)$ such that $G_i - S$ is a forest for all $i \in [\alpha]$. Here, $G_i = (V(G), E_i)$, where $E_i = \{e \in E(G) \mid i \in \mathsf{col}(e)\}$. Formally, the problem is stated below.

SIMULTANEOUS FEEDBACK EDGE SET (SIM-FES)
Input: An n-vertex graph G, $k \in \mathbb{N}$ and a coloring function $\mathsf{col} : E(G) \to 2^{[\alpha]}$.
Parameter: k, α.
Question: Is there a simultaneous feedback edge set of cardinality at most k in G?

FEEDBACK VERTEX SET (FVS) is one of the classic NP-complete [Garey and Johnson 1979] problems and has been extensively studied from all the algorithmic paradigms that are meant for coping with NP-hardness, such as approximation algorithms, parameterized complexity and moderately exponential time algorithms. The problem admits a factor 2-approximation algorithm [Bafna et al. 1999], an exact algorithm with running time $\mathcal{O}(1.7217^n n^{\mathcal{O}(1)})$ [Fomin et al. 2016], a deterministic parameterized algorithm of running in time $\mathcal{O}(3.619^k n^{\mathcal{O}(1)})$ [Kociumaka and Pilipczuk 2014], a randomized algorithm running in $\mathcal{O}(3^k n^{\mathcal{O}(1)})$ time [Cygan et al. 2011], and a kernel with $\mathcal{O}(k^2)$ vertices [Thomassé 2010]. Agrawal et al. [Agrawal et al. 2016] studied SIMULTANEOUS FEEDBACK VERTEX SET (SIM-FVS) and gave an FPT algorithm running in time $2^{\mathcal{O}(\alpha k)} n^{\mathcal{O}(1)}$ and a kernel of size $\mathcal{O}(\alpha k^{3(\alpha+1)})$. Finally, unlike the FVS problem, SIM-FES is polynomial time solvable when $\alpha = 1$, because it is equivalent to finding maximal spanning forest.

1.1. Our results and approach

In Section 3 we design an FPT algorithm for SIM-FES by reducing to α-LINEAR MATROID PARITY on the direct sum of elongated co-graphic matroids of G_i, $i \in [\alpha]$ (see Section 2 for definitions related to matroids). This algorithm runs in time $\mathcal{O}(2^{\omega k\alpha + \alpha \log k} n^{\mathcal{O}(1)})$. Unlike the vertex counterpart, we show that for $\alpha = 2$ (2-edge colored graphs) SIM-FES is polynomial time solvable. This follows from the polynomial time algorithm for the MATROID PARITY problem. In Section 4 we show that for $\alpha = 3$, SIM-FES is NP-hard. Towards this, we give a reduction from the VERTEX COVER problem in cubic graphs which is known to be NP-hard [Garey et al. 1976]. Furthermore, the same reduction shows that the problem cannot be solved in time $2^{o(k)} n^{\mathcal{O}(1)}$ unless the Exponential Time Hypothesis (ETH) fails [Impagliazzo et al. 2001]. We complement our FPT algorithms by showing that SIM-FES is W[1]-hard when parameterized by the solution size k even when $\alpha = \mathcal{O}(\log(|V(G)|))$. We show this by giving a parameter-preserving reduction from PARTITIONED HITTING SET, a variant of the HITTING SET problem, defined in [Agrawal et al. 2016]. On the other hand when $\alpha = \mathcal{O}(|V(G)|)$, we prove that the problem is in fact W[2]-hard by giving a parameter-preserving reduction from the HITTING SET problem parameterized by the solution size, a well known W[2]-hard problem [Cygan et al. 2015]. The W[1] and W[2]-hardness results are proved in Section 5. In Section 6 we give a kernel with $\mathcal{O}((k\alpha)^{\mathcal{O}(\alpha)})$ vertices. Towards this we apply some of the standard preprocessing rules for obtaining a kernel for FEEDBACK VERTEX SET and use the approach similar to the one developed for designing the kernelization algorithm for SIM-FVS [Agrawal et al. 2016]. In Section 7 we give an FPT algorithm for the problem, when parameterized by the dual parameter. Formally, this problem is defined as follows.

MAXIMUM SIMULTANEOUS ACYCLIC SUBGRAPH (MAX-SIM-SUBGRAPH)
Input: An n-vertex graph G, $q \in \mathbb{N}$ and a function $\mathsf{col} : E(G) \to 2^{[\alpha]}$.
Parameter: q.
Question: Is there a subset $F \subseteq E(G)$ such that $|F| \geq q$ and for all $i \in [\alpha]$, $G[F \cap E(G_i)]$ is acyclic?

For solving MAX-SIM-SUBGRAPH we reduce it to an equivalent instance of the α-LINEAR MATROID PARITY problem. As an immediate corollary we get an exact algorithm for SIM-FES running in time $\mathcal{O}(2^{\omega n\alpha^2} n^{\mathcal{O}(1)})$.

2. PRELIMINARIES

We denote the set of natural numbers by $\mathbb{N}$. For $n \in \mathbb{N}$, by $[n]$ we denote the set $\{1, \ldots, n\}$. For a set X, by 2^X we denote the set of all subsets of X. We use the term *ground set/ universe* to distinguish a set from its subsets. We will use ω to denote the exponent in the running time of matrix multiplication, the current best known bound for ω is < 2.373 [Williams 2012].

2.1. Graphs

We use the term *graph* to denote an undirected graph. For a graph G, by $V(G)$ and $E(G)$ we denote its vertex set and edge set, respectively. We will be considering finite graphs possibly having loops and multi-edges. In the following, let G be a graph and let H be a subgraph of G. By $d_H(v)$, we denote the degree of the vertex v in H, i.e., the number of edges in H which are incident with v. A self-loop at a vertex v contributes 2 to the degree of v. For any non-empty subset $W \subseteq V(G)$, the subgraphs of G induced by W and $V(G)\setminus W$ are denoted by $G[W]$ and $G-W$, respectively. Similarly, for $F \subseteq E(G)$, the subgraph of G induced by F is denoted by $G[F]$; its vertex set is $V(G)$ and its edge set is F. For $F \subseteq E(G)$, by $G - F$ we denote the graph obtained by deleting the edges in F. We use the convention that a double edge and a self-loop are cycles. An α-edge colored graph is a graph G with a coloring function $\mathsf{col} : E(G) \rightarrow 2^{[\alpha]}$. By G_i we will denote the color i (or i-color) graph of G, where $V(G_i) = V(G)$ and $E(G_i) = \{e \in E(G) | i \in \mathsf{col}(e)\}$. For an α-edge colored graph G, the *total degree* of a vertex v is $\sum_{i=1}^{\alpha} d_{G_i}(v)$. For example a vertex of degree 2 in G may have total degree $x + y$ where x and y are the sizes of the color sets of its two incident edges. We refer the reader to [Diestel 2012] for details on standard graph theoretic notations and terminologies.

2.2. Fields

Here we review some definitions of fields. For more details we refer to any graduate textbook on algebra. We use $\mathbb{Q}$ to denote the field on rational numbers. The number of elements in a field is called its *order*. For a prime number p, the set $\{0, 1, \ldots, p-1\}$ with addition and multiplication modulo p forms a field, which is denoted by $\mathbb{F}_p$. For every prime number p and a positive integer ℓ, there exists a unique finite field (upto isomorphism) of order p^ℓ, which is denoted by $\mathbb{F}_{p^\ell}$. For a finite field $\mathbb{F}$, $\mathbb{F}[X]$ denotes the ring of polynomials in X over $\mathbb{F}$. For the ring $\mathbb{F}[X]$, we use $\mathbb{F}(X)$ to denote the *field of fractions* of $\mathbb{F}[X]$.

2.3. Matroids

A pair $M = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a *matroid* if it satisfies the following conditions:

(I1) $\phi \in \mathcal{I}$,
(I2) if $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$, and
(I3) if $A, B \in \mathcal{I}$ and $|A| < |B|$, then there is $e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the hereditary property and a pair $(E, \mathcal{I})$ satisfying only (I2) is called hereditary family. An inclusion-wise maximal subset of $\mathcal{I}$ is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M, and is denoted by rank(M). We refer the reader to [Oxley 2006] for more details about matroids.

Representable Matroids. Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. For A, we define matroid $M = (E, \mathcal{I})$ as follows. A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are linearly independent over $\mathbb{F}$. The matroids that can be defined by such a construction are called *linear matroids*, and if a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is representable over $\mathbb{F}$. A matroid $M = (E, \mathcal{I})$ is called *representable* or *linear* if it is representable over some field $\mathbb{F}$.

Direct Sum of Matroids. Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$, $\dots$, $M_t = (E_t, \mathcal{I}_t)$ be t matroids with $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq t$. The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (E, \mathcal{I})$ with $E := \bigcup_{i=1}^{t} E_i$ and $X \subseteq E$ is independent if and only if $X \cap E_i \in \mathcal{I}_i$ for all $i \in [t]$. Let A_i be a representation matrix of $M_i = (E_i, \mathcal{I}_i)$ over field $\mathbb{F}$. Then,

$$A_M = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_t \end{pmatrix}$$

is a representation matrix of $M_1 \oplus \cdots \oplus M_t$. The correctness of this is proved in [Marx 2009; Oxley 2006].

Uniform Matroid. A matroid $M = (E, \mathcal{I})$ over an n-element ground set E, is called a uniform matroid if the family of independent sets is given by $\mathcal{I} = \{A \subseteq E \mid |A| \leq k\}$, where k is some constant. This matroid is also denoted as $U_{n,k}$.

PROPOSITION 2.1 ([OXLEY 2006]). *The uniform matroid $U_{n,k}$ is representable over any field of size strictly more than n and such a representation can be found in time polynomial in n.*

Graphic and Cographic Matroid. Given a graph G, the graphic matroid $M = (E, \mathcal{I})$ is defined by taking the edge set $E(G)$ as universe and $F \subseteq E(G)$ is in $\mathcal{I}$ if and only if $G[F]$ is a forest. Let G be a graph and η be the number of components in G. The co-graphic matroid $M = (E, \mathcal{I})$ of G is defined by taking the the edge set $E(G)$ as universe and $F \subseteq E(G)$ is in $\mathcal{I}$ if and only if the number of connected components in $G - F$ is η.

PROPOSITION 2.2 ([OXLEY 2006]). *Graphic and co-graphic matroids are representable over any field of size ≥ 2 and such a representation can be found in time polynomial in the size of the graph.*

Elongation of a Matroid. Let $M = (E, \mathcal{I})$ be a matroid and k be an integer such that $\mathsf{rank}(M) \leq k \leq |E|$. The k-elongation matroid M_k of M is the matroid with the universe as E and $S \subseteq E$ is a basis of M_k if and only if, it contains a basis of M and $|S| = k$. Observe that the rank of the matroid M_k is k.

PROPOSITION 2.3 (COROLLARY 1.2 [LOKSHTANOV ET AL. 2018]). *Let M be a linear matroid of rank r, over a ground set of size n, which is representable over a field $\mathbb{F}$. Given a number $\ell \geq r$, we can compute a representation of the ℓ-elongation of M, over the field $\mathbb{F}(X)$ in $\mathcal{O}(nr\ell)$ field operations over $\mathbb{F}$.*

Notice that in Proposition 2.3 the running time is measured in terms of the number of field operations over $\mathbb{F}$ and not over the field $\mathbb{F}(X)$. So the the number of bits in the ℓ-elongation of M is upper bounded by a polynomial function in n, r, ℓ and the number of bits in the representation of M.

PROPOSITION 2.4. *Let G be a graph with η connected components and M be an r-elongation of the co-graphic matroid associated with G, where $r \geq |E(G)| - |V(G)| + \eta$. Then $B \subseteq E(G)$ is a basis of M if and only if the subgraph $G - B$ is acyclic and $|B| = r$.*

PROOF. In the forward direction let $B \subseteq E(G)$ be a basis of M. By Definition of M it follows that $|B| = r$ and B contains a basis B_c of the co-graphic matroid of G. Suppose $G - B$ has a cycle. This implies that $G - B_c$ has a cycle. But then, there is an edge $e \in E(G - B_c)$ whose removal from $G - B_c$ does not increase the number of connected components in $G - B_c$. This contradicts that B_c was a basis in the co-graphic matroid of G.

In the reverse direction let $B \subseteq E(G)$ such that $|B| = r$ and $G - B$ is acyclic. Consider an inclusion-wise maximal subset $B' \subseteq B$ such that the number of connected components in $G - B'$ is η. Observe that $G - B'$ does not contain a cycle since $G - B$ is acyclic and B' is inclusion-wise maximal. Therefore, it follows that B' is a basis in the co-graphic matroid of G. But then B contains a basis of the co-graphic matroid of G and $|B| = r$, therefore B is a basis in M. □

α-Matroid Parity. In our algorithms we use a known algorithm for the α-LINEAR MATROID PARITY problem. Below we define α-LINEAR MATROID PARITY formally and state one of its algorithmic results.

α-LINEAR MATROID PARITY
Input: Two positive integers α and q, a linear representation A_M of a matroid $M = (E, \mathcal{I})$ and a partition $\mathcal{P}$ of E into blocks of size α.
Parameter: α, q.
Question: Does there exist an independent set which is a union of q blocks?

PROPOSITION 2.5. *There is an algorithm for α-LINEAR MATROID PARITY, running in time $\mathcal{O}(2^{\omega q \alpha}||A_M||^{\mathcal{O}(1)})$, where $||A_M||$ is the total number of bits required to describe all the elements of matrix A_M, when one of the following is true.*

(i) *The representation matrix A_M is over a field $\mathbb{F}_{p^\ell}$ or $\mathbb{Q}$.*
(ii) *Rank of $M = (E, \mathcal{I})$ is $k\alpha$ and the representation matrix A_M is over a field $\mathbb{F}(X)$, where $\mathbb{F}$ is $\mathbb{F}_{p^\ell}$ or $\mathbb{Q}$.*

Marx [Marx 2009] designed a randomized FPT algorithm for α-LINEAR MATROID PARITY, when the representation matrix A_M is over a field $\mathbb{F}_{p^\ell}$ or $\mathbb{Q}$. Lokshtanov et al. [Lokshtanov et al. 2018] derandomized it through a deterministic computation of a representation of truncation of a given linear matroid. Lokshtanov et al. proved Proposition 2.5(i) and explicitly stated it. To prove the result, in fact Lokshtanov et al. constructed a representation of $k\alpha$-trunctation of $M = (E, \mathcal{I})$ over the field $\mathbb{F}(X)$ and then solved the problem. That is, the second step of the algorithm of Lokshtanov et al. proves Proposition 2.5(ii).

3. FPT ALGORITHM FOR SIMULTANEOUS FEEDBACK EDGE SET

In this section we design an algorithm for SIM-FES by giving a reduction to α-LINEAR MATROID PARITY on the direct sum of elongated co-graphic matroids associated with graphs restricted to different color classes.

We describe our algorithm, Algo-SimFES, for SIM-FES. Let $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be an input instance to SIM-FES. Recall that for $i \in [\alpha]$, G_i is the graph with vertex set as $V(G)$ and edge set as $E(G_i) = \{e \in E(G) \mid i \in \mathsf{col}(e)\}$. Note that $n = |V(G_i)|$ for all $i \in [\alpha]$. Let η_i be the number of connected components in G_i. To make G_i acyclic we need to delete at least $|E(G_i)| - n + \eta_i$ edges from G_i. Therefore, if there is $i \in [\alpha]$ such that $|E(G_i)| - n + \eta_i > k$, then Algo-SimFES returns NO. We let $k_i = |E(G_i)| - n + \eta_i$. Observe that for $i \in [\alpha]$, $0 \leq k_i \leq k$. We need to delete at least k_i edges from $E(G_i)$ to make G_i acyclic. Therefore, the algorithm Alg-SimFES for each $i \in [\alpha]$, guesses k'_i, where $k_i \leq k'_i \leq k$ and computes a solution S of SIM-FES such that $|S \cap E(G_i)| = k'_i$. Let $M_i = (E_i, \mathcal{I}_i)$ be the k'_i-elongation of the co-graphic matroid associated with G_i.

By Proposition 2.4, for any basis F_i in M_i, $G_i - F_i$ is acyclic. Therefore, our objective is to compute $F \subseteq E(G)$ such that $|F| = k$ and the elements of F restricted to the elements of M_i form a basis for all $i \in [\alpha]$. For this we will construct an instance of α-LINEAR MATROID PARITY as follows. For each $e \in E(G)$ and $i \in \mathsf{col}(e)$, we use e^i to denote the corresponding element in M_i. For each $e \in E(G)$, by Original(e) we denote the set of ele-

ALGORITHM 1: Pseudocode of Algo-SimFES

Input: A graph $G, k \in \mathbb{N}$ and $\mathsf{col} : E(G) \to 2^{[\alpha]}$.
Output: YES if there is a simultaneous feedback edge set of size $\leq k$ and NO otherwise.

1 Let η_i be the number of connected components in G_i for all $i \in [\alpha]$
2 $k_i := |E(G_i)| - n + \eta_i$ for all $i \in [\alpha]$
3 **if** *there exists* $i \in [\alpha]$ *such that* $k_i > k$ **then**
4 | **return** NO
5 **end**
6 **for** $(k'_1, \dots, k'_\alpha) \in ([k] \cup \{0\})^\alpha$ *such that* $k_i \leq k'_i$ *for all* $i \in [\alpha]$ **do**
7 | Let M_i be the k'_i-elongation of the co-graphic matroid associated with G_i.
8 | Let $M_{\alpha+1} = U_{\tau,k'}$ over the gound set Fake(G), where, $k' = \sum_{i\in[\alpha]}(k - k'_i)$.
9 | Let $M := \bigoplus_{i\in[\alpha+1]} M_i$.
10 | For each $e \in E(G)$, let $\mathsf{Copies}(e)$ be the block of elements of M.
11 | **if** *there is an independent set of* M *composed of* k *blocks* **then**
12 | | **return** YES
13 | **end**
14 **end**
15 **return** NO

ments $\{e^j \mid j \in \mathsf{col}(e)\}$. For each edge $e \in E(G)$, and $j \in [\alpha] \setminus \mathsf{col}(e)$ we define an element e^j and a set $\mathsf{Fake}(e) = \{e^j \mid j \in [\alpha] \setminus \mathsf{col}(e)\}$. Finally, for each edge $e \in E(G)$, by $\mathsf{Copies}(e)$ we denote the set $\mathsf{Original}(e) \cup \mathsf{Fake}(e)$. Let $\mathsf{Fake}(G) = \bigcup_{e\in E(G)} \mathsf{Fake}(e)$. Furthermore, let $\tau = |\mathsf{Fake}(G)| = \sum_{e\in E(G)} |\mathsf{Fake}(e)|$ and $k' = \sum_{i\in[\alpha]}(k - k'_i)$. Let $M_{\alpha+1} = (E_{\alpha+1}, \mathcal{I}_{\alpha+1})$ be the uniform matroid of rank k' over the ground set $\mathsf{Fake}(G)$. That is, $M_{\alpha+1} = U_{\tau,k'}$. By Propositions 2.1 and Proposition 2.3 we know that M_is are representable over $\mathbb{F}_p(X)$, where $p > \max(\tau, 2)$ is a prime number and their representation can be computed in polynomial time. Let A_i be the linear representation of M_i for all $i \in [\alpha+1]$. Notice that $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq \alpha+1$. Let M denote the direct sum $M_1 \oplus \dots \oplus M_{\alpha+1}$ with its representation matrix being A_M. Note that the ground set of M is $\bigcup_{e\in E(G)} \mathsf{Copies}(e)$. Now we define an instance of α-LINEAR MATROID PARITY, which is the linear representation A_M of M and the partition of ground set into $\mathsf{Copies}(e)$, $e \in E(G)$. Notice that for all $e \in E(G)$, $|\mathsf{Copies(e)}| = \alpha$. Also for each $i \in [\alpha]$, $\mathsf{rank}(M_i) = k'_i$ and $\mathsf{rank}(M_{\alpha+1}) = k' = \sum_{i\in[\alpha]}(k - k'_i)$. This implies that $\mathsf{rank}(M) = \alpha k$.

Now Algo-SimFES outputs YES if there is a basis (an independent set of cardinality αk) of M which is a union of k blocks in M and otherwise outputs NO. Algo-SimFES uses the algorithm mentioned in Proposition 2.5 to check whether there is an independent set of M, composed of blocks. A pseudocode of Algo-SimFES can be found in Algorithm 1.

LEMMA 3.1. Algo-SimFES *is correct.*

PROOF. Let $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be a YES instance of SIM-FES and let $F \subseteq E(G)$, where $|F| = k$ be a solution of $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$. Let $k_i = |E(G_i)| - n + \eta_i$, where η_i is the number of connected components in G_i, for all $i \in [\alpha]$. For all $i \in [\alpha]$, let $k'_i = |F \cap E(G_i)|$. Since F is a solution, $k_i \leq k'_i$ for all $i \in [\alpha]$. This implies that Algo-SimFES will not execute Step 4. Consider the **for** loop for the choice $(k'_1, \dots, k'_\alpha)$. We claim that the columns corresponding to $S = \bigcup_{e\in F} \mathsf{Copies}(e)$ form a basis in M and it is union of k blocks. Note that $|S| = \alpha k$ by construction. For all $i \in [\alpha]$, let $F^i = \{e^i \mid e \in F, i \in \mathsf{col}(e)\}$, which is subset of ground set of M_i. By Proposition 2.4, for all $i \in [\alpha]$, F^i is a basis for M_i. This takes care of all the edges in $\cup_{e\in F}\mathsf{Original}(e)$. Now let $S^* = S - \cup_{i\in[\alpha]} F^i = \cup_{e\in F} \mathsf{Fake}(e)$. Observe that $|S^*| = \sum_{i\in[\alpha]}(k - k'_i) = k'$. Also, S^* is a subset of the ground set of $U_{\tau,k'}$ and thus is a basis since $|S^*| = k'$. Hence S is a basis of M. Note that S is the union of blocks corresponding to $e \in F$ and hence is union of k blocks. Therefore, Algo-SimFES will output YES.

In the reverse direction suppose Algo-SimFES outputs YES. This implies that there is a basis, say S, that is the union of k blocks. By construction S corresponds to union of the sets $\mathsf{Copies}(e)$ for some k edges in G. Let these edges be $F = \{e_1, \ldots, e_k\}$. We claim that F is a solution of $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$. Clearly $|F| = k$. Since S is a basis of M, for each $i \in [\alpha]$, $B(i) = S \cap \{e^i \mid e \in E(G_i)\}$ is a basis in M_i. Let $F(i) = \{e \mid e^i \in B(i)\} \subseteq F$. Since $B(i)$ is a basis of M_i, by Proposition 2.4, $G_i - F(i)$ is an acyclic graph. □

LEMMA 3.2. Algo-SimFES *runs in time* $\mathcal{O}(2^{\omega k\alpha+\alpha\log k}|V(G)|^{\mathcal{O}(1)})$.

PROOF. The for loop (in Step 6) runs $(k+1)^\alpha$ times. Step 11 uses the algorithm mentioned in Proposition 2.5, which takes time $\mathcal{O}(2^{\omega k\alpha}||A_M||^{\mathcal{O}(1)}) = \mathcal{O}(2^{\omega k\alpha}|V(G)|^{\mathcal{O}(1)})$. Each of the other steps in the algorithm takes polynomial time. Thus, the total running time is $\mathcal{O}(2^{\omega k\alpha+\alpha\log k}|V(G)|^{\mathcal{O}(1)})$. □

Since α-LINEAR MATROID PARITY for $\alpha = 2$ can be solved in polynomial time [Lovász 1980] algorithm Algo-SimFES runs in polynomial time for $\alpha = 2$. This gives us the following theorem.

THEOREM 3.3. SIM-FES *is in* FPT *and when* $\alpha = 2$ SIM-FES *is in* P.

4. HARDNESS RESULTS FOR SIM-FES

In this section we show that when $\alpha = 3$, SIM-FES is NP-Hard. Furthermore, from our reduction we conclude that it is unlikely that SIM-FES admits a subexponential-time algorithm. We give a reduction from VERTEX COVER (VC) in cubic graphs (i.e., every vertex has degree exactly 3) to the special case of SIM-FES where $\alpha = 3$. Let (G, k) be an instance of VC in cubic graphs, which asks whether the graph G has a vertex cover of size at most k. We assume without loss of generality that $k \le |V(G)|$. It is known that VC in cubic graphs is NP-hard [Garey et al. 1976] and unless ETH fails, it cannot be solved in time $\mathcal{O}^\star(2^{o(|V(G)|+|E(G)|)})$[1] [Johnson and Szegedy 1999]. Thus, to prove that when $\alpha = 3$, it is unlikely that SIM-FES admits a parameterized subexponential time algorithm (an algorithm of running time $\mathcal{O}^\star(2^{o(k)})$), it is sufficient to construct (in polynomial time) an instance of the form $(G', k' = \mathcal{O}(|V(G)| + |E(G)|), \mathsf{col}' : E' \to 2^{[3]})$ of SIM-FES that is equivalent to (G, k). Refer to Figure 1 for an illustration of the construction.

To construct $(G', k', \mathsf{col}' : E(G') \to 2^{[3]})$, we first construct an instance $(\widehat{G}, \widehat{k})$ of VC in subcubic graphs which is equivalent to (G, k). The graph $\widehat{G}$ is obtained from the graph G by subdividing each edge in $E(G)$ twice. Formally, we set

$$V(\widehat{G}) = V(G) \cup (\bigcup_{\{v,u\}\in E(G)} \{x_{v,u}, x_{u,v}\}), \text{ and}$$

$$E(\widehat{G}) = \bigcup_{\{v,u\}\in E(G)} \{\{v, x_{v,u}\}, \{x_{v,u}, x_{u,v}\}, \{u, x_{u,v}\}\}.$$

LEMMA 4.1. *G has a vertex cover of size k if and only if $\widehat{G}$ has a vertex cover of size* $\widehat{k} = k + |E(G)|$

[1] $\mathcal{O}^\star$ notation suppresses polynomial factors in the running-time expression.

PROOF. In the forward direction, let S be a vertex cover in G. We will construct a vertex cover $\widehat{S}$ in $\widehat{G}$ of size at most $k + |E(G)|$. Consider an edge $\{v, u\} \in E(G)$. If both u, v belongs to S, then we arbitrarily add one of the vertices from $\{x_{v,u}, x_{u,v}\}$ to $\widehat{S}$. If exactly one of $\{v, u\}$ belongs to S, say $v \in S$ then, we add $x_{u,v}$ to $\widehat{S}$. If $u \in S$, then we add $x_{v,u}$ to $\widehat{S}$. Clearly, $\widehat{S}$ is a vertex cover in $\widehat{G}$ and is of size at most $k + |E(G)|$.

In the reverse direction, given a vertex cover in $\widehat{G}$. For each $\{v, u\} \in E(G)$ such that both $x_{v,u}$ and $x_{u,v}$ are in the vertex cover, we can replace $x_{u,v}$ by u, and then, by removing all of the remaining vertices of the form $x_{v,u}$ (whose number is exactly $|E(G)|$), we obtain a vertex cover of G. □

Observe that in $\widehat{G}$ every path between two degree-3 vertices contains an edge of the form $\{x_{v,u}, x_{u,v}\}$. Thus, the following procedure results in a partition (M_1, M_2, M_3) of $E(\widehat{G})$ such that each $M_i, i \in [3]$, is a matching. Initially, $M_1 = M_2 = M_3 = \emptyset$. For each degree-3 vertex v, let $\{v, x\}$, $\{v, y\}$ and $\{v, z\}$ be the edges containing v. We insert $\{v, x\}$ into M_1, $\{v, y\}$ into M_2, and $\{v, z\}$ into M_3 (the choice of which edge is inserted into which set is arbitrary). Finally, we insert each edge of the form $\{x_{v,u}, x_{u,v}\}$ into a set M_i that contains neither $\{v, x_{v,u}\}$ nor $\{u, x_{u,v}\}$.

We are now ready to construct the instance $(G', k', \mathsf{col}' : E(G') \to 2^{[3]})$. Let $V(G') = V(\widehat{G}) \cup V^\star$, where $V^\star = \{v^\star : v \in V(\widehat{G})\}$ contains a copy $v^\star$ of each vertex v in $V(\widehat{G})$. The set $E(G')$ and coloring col' are constructed as follows. For each vertex $v \in V(\widehat{G})$, add an edge $\{v, v^\star\}$ into $E(G')$ and its color-set is $\{1, 2, 3\}$. For each $i \in [3]$ and for each $\{v, u\} \in M_i$, add the edges $\{v, u\}$ and $\{v^\star, u^\star\}$ into $E(G')$ and its color-set is $\{i\}$. We set $k' = \widehat{k}$. Clearly, the instance $(G', k', \mathsf{col}' : E(G') \to 2^{[3]})$ can be constructed in polynomial time, and it holds that $k' = \mathcal{O}(|V(G)| + |E(G)|)$.

Lemma 4.2 proves that $(\widehat{G}, \widehat{k})$ is a YES instance of VC if and only if $(G', k', \mathsf{col}' : E(G') \to 2^{[3]})$ is a YES instance of SIM-FES. Since (M_1, M_2, M_3) is a partition of $E(\widehat{G})$ and each M_i is a matching, by construction, each monochromatic cycle in G' is of the form $v - v^\star - u^\star - u - v$, where $\{v, u\} \in E(\widehat{G})$.

LEMMA 4.2. *$(\widehat{G}, \widehat{k})$ is a* YES *instance of VC if and only if $(G', k', \mathsf{col}' : E(G') \to 2^{[3]})$ is a* YES *instance of* SIM-FES.

PROOF. In the forward direction, let U be a vertex cover in $\widehat{G}$ of size at most $\widehat{k}$. Define Q as the set of edges $\{\{v, v^\star\} : v \in U\} \subseteq E(G')$. We claim that Q is a solution to $(G', k', \mathsf{col}' : E(G') \to 2^{[3]})$. Since $|Q| = |U|$, it holds that $|Q| \leq \widehat{k} = k'$. Now, consider a monochromatic cycle in G'. Recall that such a cycle is of the form $v - v^\star - u^\star - u - v$, where $\{v, u\} \in E(\widehat{G})$. Since U is a vertex cover of $\widehat{G}$, it holds that $U \cap \{v, u\} \neq \emptyset$, which implies that $Q \cap \{\{v, v^\star\}, \{u, u^\star\}\} \neq \emptyset$.

In the reverse direction, let Q be a solution to (G', k', col'). Recall that for each edge $\{v, u\} \in E(G')$, where either $v, u \in V(\widehat{G})$ or $v, u \in V^\star$ Moreover each monochromatic cycle in G' is of the form $v - v^\star - u^\star - u - v$, where $\{v, u\} \in E(\widehat{G})$. Therefore, if Q contains an edge of the form $\{v, u\}$ or of the form $\{v^\star, u^\star\}$, such an edge can be replaced by the edge $\{v, v^\star\}$. Thus, we can assume that Q only contains edges of the form $\{v, v^\star\}$. Define U as the set of vertices $\{v : \{v, v^\star\} \in Q\} \subseteq V(\widehat{G})$. We claim that U is a vertex cover of $\widehat{G}$ of size at most $\widehat{k}$. Since $|U| \leq |Q|$, it holds that $|U| \leq \widehat{k}$. Now, recall that for each edge $\{v, u\} \in E(\widehat{G})$, G' contains a monochromatic cycle of the form $v - v^\star - u^\star - u - v$. Since Q is a solution to (G', k', col'), it holds that $Q \cap \{\{v, v^\star\}, \{u, u^\star\}\} \neq \emptyset$, which implies that $U \cap \{v, u\} \neq \emptyset$. □

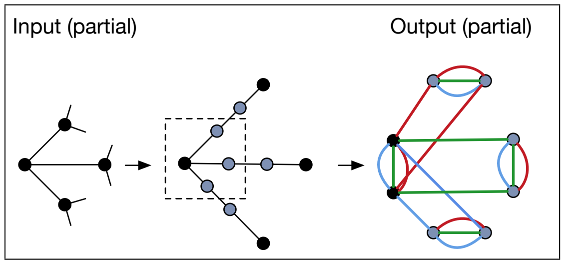

Fig. 1. The construction given in the proof of Theorem 4.3.

The following theorem is an immediate consequence of Lemma 4.1 and Lemma 4.2.

THEOREM 4.3. SIM-FES *where* $\alpha = 3$ *is* NP-hard. *Furthermore, unless the Exponential Time Hypothesis (*ETH*) fails,* SIM-FES *when* $\alpha = 3$ *cannot be solved in time* $\mathcal{O}^*(2^{o(k)})$.

5. W-HARDNESS RESULTS FOR SIMULTANEOUS FEEDBACK EDGE SET

We show that SIM-FES parameterized by k is $W[2]$ hard when $\alpha = \mathcal{O}(|V(G)|)$ and $W[1]$ hard when $\alpha = \mathcal{O}(\log(|V(G)|))$. Our reductions follow the approach of Agrawal et al. [Agrawal et al. 2016].

5.1. W[2] Hardness of SIM-FES when $\alpha = \mathcal{O}(|V(G)|)$

We give a reduction from HITTING SET (HS) to SIM-FES where $\alpha = \mathcal{O}(|V(G)|)$. Let $(U = \{u_1, \ldots, u_{|U|}\}, \mathcal{F} = \{F_1, \ldots, F_{|\mathcal{F}|}\}, k)$ be an instance of HS, where $\mathcal{F} \subseteq 2^U$, which asks whether there exists a subset $S \subseteq U$ of size at most k such that for all $F \in \mathcal{F}$, $S \cap F \neq \emptyset$. It is known that HS parameterized by k is W[2]-hard (see, e.g., [Cygan et al. 2015]). Thus, to prove the result, it is sufficient to construct (in polynomial time) an instance of the form $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ of SIM-FES that is equivalent to $(U, \mathcal{F}, k)$, where $\alpha = \mathcal{O}(|V(G)|)$. We construct a graph G such that $V(G) = \mathcal{O}(|U||\mathcal{F}|)$ and the number of colors used will be $\alpha = |\mathcal{F}|$. The intuitive idea is the following. For each element in the universe we will have an edge and this will be colored with the indices of the sets in $\mathcal{F}$, in which the element belongs to. For each $F_i \in \mathcal{F}$ we create a unique monochromatic cycle with color i which passes through all the edges corresponding to the elements it contain. Then it is easy to see that hitting monochromatic cycles in the reduced graph is equivalent to hitting all sets in $\mathcal{F}$.

Now we explain our reduction formally. Without loss of generality we assume that each set in $\mathcal{F}$ contains at least two elements from U. The instance $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ is constructed as follows. For each element $u_i \in U$, insert two new vertices into $V(G)$, v_i and w_i, add the edge $\{v_i, w_i\}$ into $E(G)$ and let $\{j \mid F_j \in \mathcal{F}, u_i \in F_j\}$ be its color-set. Now, for all $1 \leq i < j \leq |U|$ and for all $1 \leq t \leq |\mathcal{F}|$ such that $u_i, u_j \in F_t$ and $\{u_{i+1}, \ldots, u_{j-1}\} \cap F_t = \emptyset$, perform the following operation: add a new vertex into $V(G)$, $s_{i,j,t}$, add the edges $\{w_i, s_{i,j,t}\}$ and $\{s_{i,j,t}, v_j\}$ into $E(G)$ and let their color-set be $\{t\}$. Moreover, for each $1 \leq t \leq |\mathcal{F}|$, let u_i and u_j be the elements with the largest and smallest index contained in F_t, respectively, and perform the following operation: add a new vertex into $V(G)$, $s_{i,j,t}$, add the edges $\{w_i, s_{i,j,t}\}$ and $\{s_{i,j,t}, v_j\}$ into $E(G)$, and let their color-set be $\{t\}$. Observe that $|V(G)| = \mathcal{O}(|U||\mathcal{F}|)$ and that $\alpha = |\mathcal{F}|$. Therefore, $\alpha = \mathcal{O}(|V(G)|)$. It remains to show that the instances (G, k, col) and $(U, \mathcal{F}, k)$ are equivalent. By construction, each monochromatic cycle in G is of the form $v_{i_1} - w_{i_1} - s_{i_1,i_2,t} - v_{i_2} - w_{i_2} - s_{i_2,i_3,t} - \cdots - v_{i_{|F_t|}} - w_{i_{|F_t|}} - s_{i_{|F_t|},i_1,t} - v_{i_1}$, where

$\{u_{i_1}, u_{i_2}, \ldots, u_{i_{|F_t|}}\} = F_t \in \mathcal{F}$, and for each set $F_t \in \mathcal{F}$, G contains exactly one such monochromatic cycle.

LEMMA 5.1. *$(U, \mathcal{F}, k)$ is a* YES *instance of* HS *if and only if $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ is a* YES *instance of* SIM-FES.

PROOF. In the forward direction, let S be a solution to $(U, \mathcal{F}, k)$. Define Q as the set of edges $\{\{v_i, w_i\} : u_i \in S\} \subseteq E(G)$. We claim that Q is a solution to (G, k, col). Since $|Q| = |S|$, it holds that $|Q| \leq k$. Now, consider a monochromatic cycle C in G. Recall that this cycle is of the form $v_{i_1} - w_{i_1} - s_{i_1,i_2,t} - v_{i_2} - w_{i_2} - s_{i_2,i_3,t} - \cdots - v_{i_{|F_t|}} - w_{i_{|F_t|}} - s_{i_{|F_t|},i_1,t} - v_{i_1}$, where $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{|F_t|}}\} = F_t \in \mathcal{F}$. In particular, observe that $\{\{v_i, w_i\} : u_i \in F_t\} \subseteq E(C)$. Since S is is a hitting set of $\mathcal{F}$, it holds that $S \cap F_t \neq \emptyset$. This implies that $Q \cap \{\{v_i, w_i\} : u_i \in F_t\} \neq \emptyset$, and therefore Q is a solution of $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$.

In the reverse direction, let Q be a solution to $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$. By the form of each monochromatic cycle in G, if Q contains an edge that includes a vertex of the form $s_{i,j,t}$, such an edge can be replaced by the edge $\{v_i, w_i\}$. Thus, we can assume that Q only contains edges of the form $\{v_i, w_i\}$. Define S as the set of elements $\{u_i : \{v_i, w_i\} \in Q\} \subseteq U$. We claim that S is a solution to $(U, \mathcal{F}, k)$. Since $|S| \leq |Q|$, it holds that $|S| \leq k$. Now, recall that for each set $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{|F_t|}}\} = F_t \in \mathcal{F}$, G contains a monochromatic cycle of the form $v_{i_1} - w_{i_1} - s_{i_1,i_2,t} - v_{i_2} - w_{i_2} - s_{i_2,i_3,t} - \cdots - v_{i_{|F_t|}} - w_{i_{|F_t|}} - s_{i_{|F_t|},i_1,t} - v_{i_1}$. Since Q is a solution of $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$, it holds that $Q \cap \{\{v_i, w_i\} : u_i \in F_t\} \neq \emptyset$. This implies that $S \cap F_t \neq \emptyset$. □

THEOREM 5.2. SIM-FES *parameterized by k, when $\alpha = \mathcal{O}(|V(G)|)$, is $W[2]$-hard.*

5.2. W[1] Hardness of SIM-FES when $\alpha = \mathcal{O}(\log |V(G)|)$

We modify the reduction given in the proof of Theorem 5.2 to show that when $\alpha = \mathcal{O}(\log |V(G)|)$, SIM-FES is W[1]-hard with respect to the parameter k. This result implies that the dependency on α of our $\mathcal{O}((2^{\mathcal{O}(\alpha)})^k n^{\mathcal{O}(1)})$-time algorithm for SIM-FES is optimal in the sense that it is unlikely that there exists an $\mathcal{O}((2^{o(\alpha)})^k n^{\mathcal{O}(1)})$-time algorithm for this problem. More precisely, a $\mathcal{O}((2^{o(\alpha)})^k n^{\mathcal{O}(1)})$-time algorithm for SIM-FES with $\alpha = \mathcal{O}(\log n)$ will leads to an FPT algorithm for a W[1]-hard problem. Notice that when $\alpha = \mathcal{O}(\log n)$, $\mathcal{O}((2^{o(\alpha)})^k n^{\mathcal{O}(1)})$ is upper bounded by $n^{\mathcal{O}(\frac{k}{f(n)})}$ for some monotonically increasing function n. Therefore, when $k < f(n)$, $n^{\mathcal{O}(\frac{k}{f(n)})}$ is upper bounded by a polynomial function in n and when $f(n) \leq k$, the input size is upper bounded by a function in k. This will leads to an FPT-algorithm for the W[1]-hard problem we used in this section for the reduction.

We give a reduction from a variant of HS, called Partitioned Hitting Set (PHS), to SIM-FES where $\alpha = \mathcal{O}(\log |V(G)|)$. The input of PHS consists of a universe U, a collection $\mathcal{F} = \{F_1, F_2, \ldots, F_{|\mathcal{F}|}\}$, where each F_i is a family of *disjoint* subsets of U, and a parameter k. The goal is to decide the existence of a subset $S \subseteq U$ of size at most k such that for all $f \in (\bigcup_{F \in \mathcal{F}} F)$, $S \cap f \neq \emptyset$. Here we consider a special case of PHS where $|\mathcal{F}| = \mathcal{O}(\log(|U| \cdot |(\bigcup \mathcal{F})|))$. It is known that this special case is W[1]-hard when parameterized by k (see, e.g., [Agrawal et al. 2016]). Thus, to prove the theorem, it is sufficient to construct (in polynomial time) an instance of the form $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ of SIM-FES that is equivalent to $(U, \mathcal{F}, k)$, where $\alpha = \mathcal{O}(\log |V(G)|)$. The construction of the graph G is exactly similar to the one in Theorem 5.2. But instead of creating a unique monochromatic cycle with a color i for each $f_i \in \bigcup \mathcal{F}$, for each $F_i \in \mathcal{F}$ we create $|F_i|$ vertex disjoint cycles of same color i. Since for each $F \in \mathcal{F}$ the sets in F are pairwise disjoint, guarantees the correctness. Formal description of the reduction is given below.

Without loss of generality we assume that each set in $\bigcup_{F\in\mathcal{F}} F$ contains at least two elements from U and each element in U is present in some set in $(\bigcup_{F\in\mathcal{F}} F)$. The instance $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ is constructed as follows. For each element $u_i \in U$, insert two new vertices v_i and w_i into $V(G)$, and add the edge $\{v_i, w_i\}$ into $E(G)$ with its color-set being $\{j : F_j \in \mathcal{F}, u_i \in (\bigcup F_j)\}$. Now, for all $1 \le i < j \le |U|$ and for all $1 \le t \le |\mathcal{F}|$ such that there exists $f \in F_t$ satisfying $u_i, u_j \in f$ and $\{u_{i+1}, \ldots, u_{j-1}\} \cap f = \emptyset$, perform the following operation: add a new vertex $s_{i,j,t}$ into $V(G)$, add the edges $\{w_i, s_{i,j,t}\}$ and $\{s_{i,j,t}, v_j\}$ into $E(G)$ with both of its color-set being $\{t\}$. Moreover, for each $1 \le t \le |\mathcal{F}|$ and $f \in F_t$, let u_i and u_j be the elements with the largest and smallest index contained in f, respectively, we perform the following operation: add a new vertex into $V(G)$, $s_{i,j,t}$, add the edges $\{w_i, s_{i,j,t}\}$ and $\{s_{i,j,t}, v_j\}$ into $E(G)$, and let their color-set be $\{t\}$. Observe that $|V(G)| = \Omega(|U| + |(\bigcup \mathcal{F})|)$ and that $\alpha = |\mathcal{F}|$. Since $|\mathcal{F}| = \mathcal{O}(\log(|U||(\bigcup \mathcal{F})|))$, we have that $\alpha = \mathcal{O}(\log |V(G)|)$. Since the sets in each family F_i are disjoint, the construction implies that each monochromatic cycle in G is of the form $v_{i_1} - w_{i_1} - s_{i_1,i_2,t} - v_{i_2} - w_{i_2} - s_{i_2,i_3,t} - \cdots - v_{i_{|f|}} - w_{i_{|f|}} - s_{i_{|f|},i_1,t} - v_{i_1}$, where $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{|F_t|}}\} = f$ for a set $f \in F_t \in \mathcal{F}$, and for each set $f \in F_t \in \mathcal{F}$, G contains a monochromatic cycle of this form. By using the arguments similar to one in the proof of Lemma 5.1, we get that the instances $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ and $(U, \mathcal{F}, k)$ are equivalent. Hence we get the following theorem.

THEOREM 5.3. SIM-FES *parameterized by k, when* $\alpha = \mathcal{O}(\log |V(G)|)$ *is* $W[1]$*-hard.*

6. KERNEL FOR SIMULTANEOUS FEEDBACK EDGE SET

In this section we give a kernel for SIM-FES with $\mathcal{O}((k\alpha)^{\mathcal{O}(\alpha)})$ vertices. We start by applying preprocessing rules similar in spirit to the ones used to obtain a kernel for FEEDBACK VERTEX SET, but it requires subtle differences due to the fact that we handle a problem where edges rather than vertices are deleted, as well as the fact that the edges are colored (in particular, each edge in SIM-FES has a color-set, while each vertex in SIM-FVS is uncolored). We obtain an approximate solution by computing a spanning tree per color. We rely on the approximate solution to bound the number of vertices whose degree in certain subgraphs of G is not equal to 2. Then, the number of the remaining vertices is bounded by adapting the “interception”-based approach of Agrawal et al. [Agrawal et al. 2016] to a form relevant to SIM-FES.

Let $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be an instance of SIM-FES. Recall that for each color $i \in [\alpha]$, G_i is the graph consisting of the vertex-set $V(G)$ and the edge-set $E(G_i)$ includes every edge in $E(G)$ whose color-set contains the color i. It is easy to verify that the following rules are correct when applied exhaustively in the order in which they are listed. We note that the resulting instance can contain multiple edges.

— **Reduction Rule 1:** If $k < 0$, return that $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ is a NO instance.
— **Reduction Rule 2:** If for all $i \in [\alpha]$, G_i is acyclic, return that $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ is a YES instance.
— **Reduction Rule 3:** If there is a self-loop at a vertex $v \in V(G)$, then remove v from G and decrement k by 1.
— **Reduction Rule 4:** If there exists an isolated vertex in G, then remove it.
— **Reduction Rule 5:** If there exists $i \in [\alpha]$ and an edge whose color-set contains i but it does not participate in any cycle in G_i, remove i from its color-set. If the color-set becomes empty, remove the edge.
— **Reduction Rule 6:** If there exists $i \in [\alpha]$ and a vertex v of degree exactly two in G, remove v and connect its two neighbors by an edge whose color-set is the same as the color-set of the two edges incident to v (we prove in Lemma 6.1 that the color sets of the two edges are same).

LEMMA 6.1. *Reduction rule 6 is safe.*

PROOF. Let G be a graph with coloring function $\mathsf{col} : E(G) \to 2^{[\alpha]}$. Let v be a vertex in $V(G)$ such that v has total degree 2 in G. We have applied Reduction Rule 1 to 5 exhaustively (in that order). Therefore, when Rule 6 is applied, the edges incident to v have the same color-set (say Q), since otherwise Rule 5 would be applicable. Let u, w be the neighbors of v in G. Consider the graph G' with vertex set as $V(G) \setminus \{v\}$ and edge set as $E(G') = (E(G) \setminus \{\{v, u\}, \{v, w\}\}) \cup \{e_{uw} = \{u, w\}\}$ (we add a new edge e_{uw} even if there exist one or more edges between u and w in G). We define a coloring function col' such that $\mathsf{col}'(e_{uw}) = Q$ and for all other edges $e \in E(G') \setminus \{e_{uw}\}$, $\mathsf{col}'(e) = \mathsf{col}(e)$. We show that $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ is a YES instance of SIM-FES if and only if (G', k, col') is a YES instance of SIM-FES.

In the forward direction, let S be a solution to SIM-FES in G of size at most k. Suppose S is not a solution in G'. Then, there is a cycle C in G'_t, for some $t \in [\alpha]$. Note that C cannot be a cycle in G'_j as $G'_j = G_j$, for $j \in [\alpha] \setminus Q$. Therefore C must be a cycle in G'_i for some $i \in Q$. All the cycles C' not containing the edge e_{uw} are also cycles in G_i and therefore S must contain some edge from C'. It follows that C must contain the edge e_{uw}. Note that the edges $(E(C) \setminus \{e_{uw}\}) \cup \{\{v, u\}, \{w, v\}\}$ form a cycle in G_i. Therefore S must contain an edge from $E(C) \cup \{\{v, u\}, \{w, v\}\}$. We consider the following cases:

— Case 1: $\{v, u\}, \{w, v\} \notin S$. In this case S must contains an edge from $E(C) \setminus \{\{u, w\}\}$. Hence, S is a solution in G'.
— Case 2: At least one of $\{v, u\}, \{w, v\}$ belongs to S, say $\{v, u\} \in S$. Let $S' = (S \setminus \{\{v, u\}\}) \cup \{e_{uw}\}$. Observe that S' intersects all cycles in G'_i. Therefore S' is a solution in G' of size at most k.

In the reverse direction, consider a solution S' to SIM-FES in G'. We construct a solution S to SIM-FES in G' as follows. If $e_{uw} \notin S$, then $S' = S$. Otherwise $S = (S \setminus \{e_{uw}\}) \cup \{u, w\}$. If S is a solution in G we have a proof of the claim. Therefore, for the sake of contradiction we assume that S is not a solution in G. Notice that S intersects all cycles in G_j, for all $j \in [\alpha] \setminus Q$ (since $G_j = G'_j$ and $e_{uw} \notin E(G_j)$). Also notice that for all $i \in Q$, all cycles in G_i not containing v are also cycles in G'_i and therefore S intersects all such cycles. If S does not hit a cycle C in G_i, then $i \in Q$ and C must contain uvw as a subpath (because v is a degree-two vertex in G). Note that in G'_i we added an edge e_{uw} and we keep multi-edges. There S' must contain the edge e_{uw}. This implies that $\{u, v\} \in S$. This contradicts the assumption that S does not hit C. This completes the proof. □

We apply Reduction Rule 1 to 6 exhaustively (in that order). The safeness of Reduction Rules 1 to 5 are easy to see. Lemma 6.1 proves the safeness Reduction Rule 6. After this, we follow an approach similar to that in [Agrawal et al. 2016] to bound the size of the instance. This gives the following theorem.

THEOREM 6.2. SIM-FES *admits a kernel with* $(k\alpha)^{\mathcal{O}(\alpha)}$ *vertices.*

PROOF. Let $(G, k, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be an instance of SIM-FES where none of the Reduction rules are applicable. For each graph G_i, we compute a spanning forest, F_i, maximizing $|E(F_i)|$. Let $X_i = E(G_i) \setminus E(F_i)$. If $|X_i| > k$, the instance is a no-instance. Thus, we can assume that for each $i \in [\alpha]$, X_i contains at most k edges. Let $X = \bigcup_{i=1}^{\alpha} X_i$ denote the union of the sets X_i. Clearly, $|X| \leq k\alpha$. Let U denote the subset of $V(G)$ that contains the vertices incident to at least one edge in X. Since Reduction Rule 5 is not applicable, therefore $|U| \leq 2k\alpha$. Thus, the number of leaves in each $G_i - X$ is bounded by $2k\alpha$. Accordingly, the number of vertices in each $G_i - X$ whose degree is at least 3 is bounded by $2k\alpha$. It remains to bound the number of vertices that are not incident to any edge in X and whose degree in each G_i is 0 or 2 (their degree in G is at least 3). Let T be the set of vertices in G which is either a leaf or a degree 3 vertex in some G_i, for $i \in [\alpha]$. Denote the set of vertices which are not in T, not incident to any edge in X and whose degree in G_i is 2 by D_i. Let $\mathcal{P}_i$ denote the set of paths in G_i, for $i \in [\alpha]$, whose internal vertices belong to D_i and whose first and last vertices do not belong to D_i. Moreover, let $D = \bigcup_{i=1}^{\alpha} D_i$ and $\mathcal{P} = \bigcup_{i=1}^{\alpha} \mathcal{P}_i$. Observe that for $i \in [\alpha]$, $|\mathcal{P}_i| \leq 4k\alpha$ and $|\mathcal{P}| \leq 4k\alpha^2$.

Now we prove that $|D| = \mathcal{O}((k\alpha)^{\mathcal{O}(\alpha)})$. For each edge $e \in E(G)$, let $\mathcal{P}[e]$ be the set of paths in $\mathcal{P}$ to which e belongs. Each edge belongs to at most one path in each $\mathcal{P}_i$, for any $i \in [\alpha]$. For each $v \in D$, by $E(v)$ we denote the set of edges incident to v in G. Observe that each vertex in D is incident to at most 2α edges. For each vertex $v \in D$, there are at most $(4k\alpha + 1)^{\alpha}$ options of choosing to which paths in $\mathcal{P}$ the vertex v belongs. Note that here the extra additive one is to include the case when a vertex does not belong to any path in a color class. Thus, there exists a constant c such that if $|D| > (k\alpha)^{c\alpha}$, then D contains (at least) three vertices, r, s and t, such that for all $q, p \in \{r, s, t\}$, there is a bijection $f : E(q) \to E(p)$ such that $\mathcal{P}[e] = \mathcal{P}[f(e)]$ for all $e \in E(q)$. In particular, if $|D| > (k\alpha)^{c\alpha}$, then D contains two non-adjacent vertices, v and u, such that there is a bijection $f : E(v) \to E(u)$ satisfying $\mathcal{P}[e] = \mathcal{P}[f(e)]$ for all $e \in E(v)$. In this case, it is not necessary to insert any edge $e \in E(v)$ into a solution, since it has the same affect as inserting the edge $f(e)$. Thus, we can remove the vertex v, and for each two neighbors of v, x and y, and for each color $i \in [\alpha]$ such that $i \in \mathsf{col}(\{v, x\}) \cap \mathsf{col}(\{v, y\})$, we insert an edge $\{x, y\}$ whose color-set is $\{i\}$. After an exhaustive application of this operation (as well as Reduction Rules 1–6), we obtain the desired bound on $|D|$, which concludes the proof of Theorem 6.2. $\square$

7. MAXIMUM SIMULTANEOUS ACYCLIC SUBGRAPH

In this section we design an algorithm for MAXIMUM SIMULTANEOUS ACYCLIC SUBGRAPH. Let $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be an input to MAX-SIM-SUBGRAPH. A set $F \subseteq E(G)$ such that for all $i \in [\alpha]$, $G[F_i]$ is acyclic is called *simultaneous forest*. Here, $F_i = \{e \in F \mid i \in \mathsf{col}(e)\}$, denotes the subset of edges of F which has the integer i in its image when the function col is applied to it. We will solve MAX-SIM-SUBGRAPH by reducing to an equivalent instance of the α-LINEAR MATROID PARITY problem and then using the algorithm for the same.

We start by giving a construction that reduces MAX-SIM-SUBGRAPH to α-LINEAR MATROID PARITY. Let $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be an input to MAX-SIM-SUBGRAPH. Given, $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$, for $i \in [\alpha]$, recall that by G_i we denote the graph with the vertex set $V(G_i) = V(G)$ and the edge set $E(G_i) = \{e^i \mid e \in E(G) \text{ and } i \in \mathsf{col}(e)\}$. For each edge $e \in E(G)$, we will have its distinct copy in G_i if $i \in \mathsf{col}(e)$. Thus, for each edge $e \in E(G)$, by $\mathsf{Original}(e)$ we denote the set of edges $\{e^j \mid j \in \mathsf{col}(e)\}$. On the other hand for each edge $e \in E(G)$, by $\mathsf{Fake}(e)$ we denote the set of edges $\{e^j \mid j \in [\alpha] \setminus \mathsf{col}(e)\}$. Finally, for each edge $e \in E(G)$, by $\mathsf{Copies}(e)$ we denote the set $\mathsf{Original}(e) \cup \mathsf{Fake}(e)$. Let $M_i = (E_i, \mathcal{I}_i)$ denote the graphic matroid on G_i. That is, the edges of G_i form universe E_i and $\mathcal{I}_i$ contains, $S \subseteq E(G_i)$ such that $G_i[S]$ forms a forest. By Proposition 2.2 we know that graphic matroids are representable over any field and given a graph G one can find the corresponding representation matrix in time polynomial in $|V(G)|$. Let A_i denote the linear representation of M_i. That is, A_i is a matrix over $\mathbb{F}_2$, where the set of columns of A_i are denoted by $E(G_i)$. In particular, A_i has dimension $d \times |E(G_i)|$, where $d = \mathsf{rank}(M_i)$. A set $X \subseteq E(G_i)$ is independent (that is $X \in \mathcal{I}_i$) if and only if the corresponding columns are linearly independent over $\mathbb{F}_2$. Let $\mathsf{Fake}(G)$ denote the set of edges in $\bigcup_{e \in E(G)} \mathsf{Fake}(e)$. Furthermore, let $\tau = |\mathsf{Fake}(G)| = \sum_{e \in E(G)} |\mathsf{Fake}(e)|$. Let $M_{\alpha+1}$ be the uniform matroid over $\mathsf{Fake}(G)$ of rank τ. That is, $E_{\alpha+1} = \mathsf{Fake}(G)$

and $M_{\alpha+1} = U_{\tau,\tau}$. Let I_τ denote the identity matrix of dimension $\tau \times \tau$. Observe that, $A_{\alpha+1} = I_\tau$ is a linear representation of $M_{\alpha+1}$ over $\mathbb{F}_2$. Notice that $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq \alpha+1$. Let M denote the direct sum of $M_1 \oplus \cdots \oplus M_{\alpha+1}$ with its representation matrix being A_M.

Now we are ready to define an instance of α-LINEAR MATROID PARITY. The ground set is the columns of A_M, which is indexed by edges in $\bigcup_{e \in E(G)} \mathsf{Copies}(e)$. Furthermore, the ground set is partitioned into $\mathsf{Copies}(e)$, $e \in E(G)$, which are called blocks. The main technical lemma of this section on which the whole algorithm is based is the following.

LEMMA 7.1. *Let* $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ *be an instance of* MAX-SIM-SUBGRAPH. *Then* G *has a simultaneous forest of size* q *if and only if* $(A_M, \biguplus_{e \in E(G)} \mathsf{Copies}(e), q)$ *is a* YES *instance of* α-LINEAR MATROID PARITY. *Furthermore, given* $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ *we can obtain an instance* $(A_M, \biguplus_{e \in E(G)} \mathsf{Copies}(e), q)$ *in polynomial time.*

PROOF. We first show the forward direction of the proof. Let F be a simultaneous forest of size q. Then we claim that the columns corresponding to $S = \bigcup_{e \in F} \mathsf{Copies}(e)$ form an independent set in M and furthermore, it is the union of q blocks. That is, we need to show that the columns corresponding to $S = \bigcup_{e \in F} \mathsf{Copies}(e)$ are linearly independent in A_M over $\mathbb{F}_2$. By the definition of direct sum and its linear representation, it reduces to showing that F is linearly independent if and only if $F \cap E_i \in \mathcal{I}_i$ for all $i \leq \alpha + 1$. Since F is a simultaneous forest of size q, we have that $G[F_i]$, $F_i = \{e \in F \mid i \in \mathsf{col}(e)\}$, is a forest. Hence, this implies that $Q_i = \{e^i \mid e \in F_i\}$ forms a forest in G_i. This takes care of all the edges in $\cup_{e \in F} \mathsf{Original}(e)$. Now let $S^* = S \setminus (\bigcup_{i \in [\alpha]} Q_i) = \cup_{e \in F} \mathsf{Fake}(e) = Q_{\alpha+1}$. However, S^* is a subset of $U_{\tau,\tau}$ and thus is an independent set since $|S^*| \leq \tau$. This completes the proof of the forward direction.

Now we show the reverse direction of the proof. Since, $(A_M, \biguplus_{e \in E(G)} \mathsf{Copies}(e), q)$ is a yes instance of α-LINEAR MATROID PARITY, there exists an independent set, say S, that is the union of q blocks. By construction S corresponds to union of the sets $\mathsf{Copies}(e)$ for some q edges in G. Let these edges be $F = \{e_1, \ldots, e_q\}$. We claim that F is a simultaneous forest of size q. Towards this, we need to show that $G[F_i]$, where $F_i = \{e \in F \mid i \in \mathsf{col}(e)\}$, is a forest. This happens if and only if $Q_i = \{e^i \mid e \in F_i\}$ forms a forest in G_i. However, we know that the columns corresponding to Q_i are linearly independent in M_i and in particular in A_i – the linear representation of graphic matroid of G_i. This shows that Q_i forms a forest in G_i and hence $G[F_i]$ is a forest. This completes the equivalence proof.

Finally, it easily follows from the discussion preceding the lemma that given $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ we can obtain an instance $(A_M, \biguplus_{e \in E(G)} \mathsf{Copies}(e), q)$ in time polynomial in $|V(G)|$. This completes the proof of the lemma. $\square$

Given an instance $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ of MAX-SIM-SUBGRAPH we first apply Lemma 7.1 and obtain an instance $(A_M, \biguplus_{e \in E(G)} \mathsf{Copies}(e), q)$ of α-LINEAR MATROID PARITY and then apply Proposition 2.5 to obtain the following result.

THEOREM 7.2. MAX-SIM-SUBGRAPH *can be solved in time* $\mathcal{O}(2^{\omega q \alpha} |V(G)|^{\mathcal{O}(1)})$.

Let $(G, q, \mathsf{col} : E(G) \to 2^{[\alpha]})$ be an instance of MAX-SIM-SUBGRAPH. Observe that q is upper bounded by $\alpha(|V(G)| - 1)$. Thus, as a corollary to Theorem 7.2 we get an exact algorithm for finding the largest sized simultaneous acyclic subgraph, running in time $\mathcal{O}(2^{\omega n \alpha^2} |V(G)|^{\mathcal{O}(1)})$.

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