A Polynomial Kernel for Deletion to Ptolemaic Graphs

Akanksha Agrawal
Indian Institute of Technology Madras, Chennai, India

Aditya Anand
Indian Institute of Technology Kharagpur, Kharagpur, India

Saket Saurabh
Institute of Mathematical Sciences, Chennai, India
University of Bergen, Bergen, Norway

Abstract

For a family of graphs $\mathcal{F}$, given a graph G and an integer k, the $\mathcal{F}$-Deletion problem asks whether we can delete at most k vertices from G to obtain a graph in the family $\mathcal{F}$. The $\mathcal{F}$-Deletion problems for all non-trivial families $\mathcal{F}$ that satisfy the hereditary property on induced subgraphs are known to be NP-hard by a result of Yannakakis (STOC'78). Ptolemaic graphs are the graphs that satisfy the Ptolemy inequality, and they are the intersection of chordal graphs and distance-hereditary graphs. Equivalently, they form the set of graphs that do not contain any chordless cycles or a gem as an induced subgraph. (A gem is the graph on 5 vertices, where four vertices form an induced path, and the fifth vertex is adjacent to all the vertices of this induced path.) The Ptolemaic Deletion problem is the $\mathcal{F}$-Deletion problem, where $\mathcal{F}$ is the family of Ptolemaic graphs. In this paper we study Ptolemaic Deletion from the viewpoint of Kernelization Complexity, and obtain a kernel with $\mathcal{O}(k^6)$ vertices for the problem.

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases Ptolemaic Deletion, Kernelization, Parameterized Complexity, Gem-free chordal graphs

Digital Object Identifier 10.4230/LIPIcs...

Funding *Akanksha Agrawal*: Supported by New Faculty Initiation Grant (IIT Madras).
Saket Saurabh: Supported by funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 819416) and Swarnajayanti Fellowship grant DST/SJF/MSA-01/2017-18.

© Akanksha Agrawal, Aditya Anand and Saket Saurabh;
licensed under Creative Commons License CC-BY 4.0

1 Introduction

Graph modification problems are one of the central problems in Graph Theory, and vertex deletion is one of the most natural graph modification operations. For a family of graphs $\mathcal{F}$, $\mathcal{F}$-Deletion takes as input an n-vertex graph G and an integer k, and the objective is to determine whether we can remove a set of at most k vertices from G to obtain a graph in $\mathcal{F}$. Some of the classical examples of $\mathcal{F}$-Deletion are the NP-hard problems like Vertex Cover, Feedback Vertex Set, and Odd Cycle Transversal, corresponding to $\mathcal{F}$ being the family of edgeless graphs, forests, and bipartite graphs, respectively. Unfortunately, most of these $\mathcal{F}$-Deletion problems are NP-hard by a result of [19, 22]. Thus, they have received substantial attention in the algorithmic paradigms for coping with NP-hardness, including Approximation Algorithms and Parameterized Complexity.

In Parameterized Complexity each problem instance is accompanied by a parameter k. One of the central notions in parameterized complexity is that of *fixed parameter tractability* (FPT). A parameterized problem Π is FPT if given an instance (I, k) of Π, we can determine whether (I, k) is a yes-instance of Π in time bounded by $\mathcal{O}(f(k)|I|^{\mathcal{O}(1)})$, where f is some computable function of k and $|I|$ is the encoding length of I. One way to obtain an FPT algorithm for a (decidable) parameterized problem algorithm is to exhibit a *kernelization algorithm*, or *kernel*. A kernel for a problem Π is an algorithm that given an instance (I, k) of Π, runs in polynomial time and outputs an equivalent instance (I', k') of Π such that $|I'|, k'$ are both upper bounded by $g(k)$ for some computable function g. The function g is the *size* of the kernel, and if g is a polynomial function, then we say that the kernel is a *polynomial kernel*. A kernel for a decidable problem implies that it admits an FPT algorithm, but kernels are also very interesting in their own right, as they mathematically capture the efficiency of polynomial time pre-processing routines.

A graph is a *chordal* graph if it does not contain any induced cycle on at least four vertices. A graph G is *distance hereditary* if the distances between vertices in every connected induced subgraph of G are the same as in the graph G. *Ptolemaic graphs* are the graphs that are both chordal and distance hereditary. In this paper we study the $\mathcal{F}$-Deletion problem corresponding to the family of Ptolemaic graphs. Formally, we study the following problem.

Ptolemaic Deletion	**Parameter:** k
Input: A graph G with n vertices and a non-negative integer k.	
Question: Is there $X \subseteq V(G)$ such that $|X| \leqslant k$ and $G \setminus X$ is a Ptolemaic graph?	

We study the parameterized complexity of the Ptolemaic Deletion problem. Recently, Ahn et al. [7] obtained a constant-factor approximation algorithm for the weighted version of the problem. In their algorithm, for a given instance of Ptolemaic Deletion, they create an equivalent (and approximation preserving) instance of a special Feedback Vertex Set problem, which they call Feedback Vertex Set with Precedence Constraints. With a minor modification to the well-known iterative-compression based FPT algorithm for Feedback Vertex Set (see, for example [9]), we can obtain a $c^k n^{\mathcal{O}(1)}$-time algorithm for Feedback Vertex Set with Precedence Constraints. The above together with the reduction from Ptolemaic Deletion to the problem Feedback Vertex Set with Precedence Constraints presented in [7] implies that Ptolemaic Deletion admits a simple $c^k n^{\mathcal{O}(1)}$-time FPT algorithm. We remark that both Chordal Vertex Deletion [20] and Distance Hereditary Deletion [13] are FPT, but their algorithms are more involved, compared to the elegant FPT algorithm that Ptolemaic Deletion admits.

Due to the existence of a simple single-exponential FPT algorithm for Ptolemaic

Deletion, in this paper we focus on obtaining a polynomial kernel for the problem. In particular, we obtain the following result.

▶ **Theorem 1.** Ptolemaic Deletion *admits a kernel with* $\mathcal{O}(k^6)$ *vertices.*

We note that the kernelization complexity of Chordal Vertex Deletion was a well-known open problem in the field of Parameterized Complexity. Jansen and Pilipczuk [16] designed the first polynomial kernel for the above problem, and shortly afterwards, Agrawal et al. [3] gave an improved kernel with $\mathcal{O}(k^{13})$ vertices. Kim and Kwon [17] showed that Distance Hereditary Deletion admits a kernel of size $\mathcal{O}(k^{30} \log^5 k)$. We remark that the kernel we obtain has only $\mathcal{O}(k^6)$ vertices, which is much smaller than the kernels for both these problems. We also believe that our techniques can prove to be useful in obtaining/improving kernels for related classes of graphs.

Our Methods: We now describe the techniques we use in obtaining our kernel, for the given instance (G, k) of Ptolemaic Deletion. First, we compute an approximate solution S, using the result of Ahn et al. [7]. We further construct a strengthened version of the approximate solution called a redundant solution, D, introduced in [4]. Roughly speaking, redundant solutions allow us to, in a sense, forget about small obstructions. This simplifies many technical difficulties, while only maintaining a small set of vertices. We analyse the set of maximal cliques and bound the size of any maximal clique in the Ptolemaic graph $G \setminus (S \cup D)$ by $\mathcal{O}(k^3)$ by exploiting structural properties of Ptolemaic graphs and the property of "redundancy" ensured by the set D. To this end, we also use matchings in auxiliary graphs, to determine which vertices are "safe" to delete.

After bounding the sizes of maximal cliques in $G \setminus (S \cup D)$, we make use of a characterization of Ptolemaic graphs based on their inter-clique digraph [21]. In particular, Uehara and Uno [21] showed that a graph is Ptolemaic if and only if the underlying undirected graph of the inter-clique digraph of the given graph is a forest. We now use the concept of independence degree introduced in [3], and bound the independence degree of certain vertices, introducing an annotated version of the problem, similar to [3]. We then design a procedure which exploits the bounded independence degree and obtain a set $R \subseteq D$, so that the number of leaves in the undirected inter-clique digraph of the Ptolemaic graph $G \setminus (R \cup S)$ can be bounded by $\mathcal{O}(k^3)$. Every vertex of this forest, which we call bags, corresponds to a set of vertices which form a clique in $G \setminus (R \cup S)$. This, together with the fact that the size of a maximal clique is bounded in $G \setminus (R \cup S)$, gives us a bound on the number of vertices contained in degree-1 and degree $\geqslant 3$ bags in the (undirected) forest. Finally, we bound the number of vertices in degree-2 bags. In order to do this, we examine the structure of degree-2 paths (paths containing only degree 2 bags of the forest) in the inter-clique digraph. We first give several structural reduction rules and then show that we can safely replace "large" portions of the graph with smaller-sized graphs, if we maintain the size of a minimum separator in an augmented graph. Combining everything together, we obtain our kernel for Ptolemaic Deletion with $\mathcal{O}(k^6)$ vertices.

Related Works: As Ptolemaic graphs are hereditary on induced subgraphs, by the seminal result of Lewis and Yannakakis [22, 19] it follows that Ptolemaic Deletion is NP-complete. As mentioned earlier, very recently Ahn et al. [7] obtained a constant factor approximation algorithm for Ptolemaic Deletion. There have been several studies of the parameterized complexity for deletion to subclasses of chordal graphs and distance hereditary graphs. As mentioned previously, both Chordal Vertex Deletion [3, 20, 16] and Distance Hereditary Deletion [13, 17] admit FPT algorithms and polynomial kernels. Interval graphs are an important subclass of chordal graphs. Cao and Marx [8] showed that Interval Vertex Deletion admits an FPT algorithm. Recently, Agrawal et al. [4] obtained the first polynomial kernel for Interval Vertex Deletion with $\mathcal{O}(k^{1741})$ vertices. Block graphs are a subclass of Ptolemaic graphs, and Block Vertex Deletion is known to admit an FPT algorithm running in time $4^k n^{\mathcal{O}(1)}$ and a kernel with $\mathcal{O}(k^4)$ vertices [2]. Split graphs form a well-known subclass of chordal graphs, and Split Vertex Deletion is known to admit a $\mathcal{O}(1.2738^k k^{\mathcal{O}(\log k)} + n^{\mathcal{O}(1)})$-time FPT algorithm [10] and a kernel with $\mathcal{O}(k^2)$ vertices [1]. Another class of graphs which form a subclass of Ptolemaic graphs are 3-leaf power graphs. The corresponding deletion problem, 3-Leaf Power Deletion was shown to be FPT by [11, 5]. Recently, Ahn et al. [6] designed a polynomial kernel for 3-Leaf Power Deletion.

2 Preliminaries

Sets and Undirected Graphs. For $k \in \mathbb{N}$, we use $[k]$ as a shorthand for $\{1, 2, \ldots, k\}$. Given an undirected graph G, we let $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. We let n denote the number of vertices in a graph G, whenever the context is clear. The *open neighborhood*, or simply the *neighborhood*, of a vertex $v \in V(G)$ is defined as $N_G(v) = \{w \mid \{v, w\} \in E(G)\}$. We extend the definition of neighborhood of a vertex to a set of vertices as follows. Given a subset $U \subseteq V(G)$, $N_G(U) = \bigcup_{u \in U} N_G(u) \setminus U$. We omit subscripts when the graph G is clear from the context. The *induced subgraph* $G[U]$ is the graph with vertex-set U and edge set $\{\{u, u'\} \mid u, u' \in U, \text{ and } \{u, u'\} \in E(G)\}$. Moreover, we define $G \setminus U$ as the induced subgraph $G[V(G) \setminus U]$. An *independent set* in G is a set of vertices such that there is no edge between any pair of vertices in this set. A *clique* in G is a set of vertices such that there is an edge between every pair of distinct vertices in this set. A *matching* in G is a set of edges, no two of which share a common vertex. A matching with the largest possible number of edges is called a *maximum matching*.

A *path* P in a graph G is a subgraph of P with vertex set $V(P) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(P) = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{\ell-1}, x_\ell\}\} \subseteq E(G)$. The length of a path is the number of vertices in its vertex set. The vertices x_1 and x_ℓ are the *endpoints* of P and the remaining vertices in $V(P)$ are its *internal vertices*. We also say that P is a path between x_1 and x_ℓ. A *cycle* C in G is a subgraph with vertex set $V(C) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(C) = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{\ell-1}, x_\ell\}, \{x_\ell, x_1\}\} \subseteq E(G)$. The length of a cycle is the number of vertices in its vertex set.

A *spanning cycle* of a graph G is a cycle C in G, such that $V(C) = V(G)$. Let P be a path in a graph G on at least three vertices. We say that $\{u, v\} \in E(G)$ is a *chord* if $u, v \in V(P)$ but $\{u, v\} \notin E(P)$. Similarly, for a cycle C on at least four vertices, $\{u, v\} \in E(G)$ is a chord of C if $u, v \in V(C)$ but $\{u, v\} \notin E(C)$. A path P or cycle C in G is *chordless* if it has no chords. Let us note that any chordless cycle has length at least 4.

The graph G is *connected* if there is a path between every pair of distinct vertices, otherwise G is *disconnected*. A connected graph without any cycles is a *tree*, and a collection of trees is a *forest*. A (vertex) inclusion-wise maximal connected induced subgraph of G is called a *connected component* of G. We define the distance $d(u, v)$ between two vertices $u, v \in V(G)$ as the length of the shortest path between them.

For a graph G and vertices $s, t \in V(G)$, where $s \neq t$, an *s-t separator* is a subset $U \subseteq V(G)$ such that $G \setminus U$ has no s-t path. It is well known that a minimum sized s-t separator (also called a *minimum s-t separator*) can be found in polynomial time using, for example, an algorithm for maximum flow, see for instance Chapter 7 of [18].

A *gem* is a graph on five vertices, with one vertex adjacent to each of the remaining four vertices which form an induced path.

A graph G is *chordal* if it does not contain a chordless cycle as an induced subgraph. G is *distance hereditary* if for every connected induced subgraph H of G and two distinct vertices $u, v \in V(H)$, the length of the shortest path between u and v in H is equal to the length of the shortest path between u and v in G.

A graph G is *bipartite* if $V(G)$ can be partitioned into two sets X and Y, so that every edge in $E(G)$ has exactly one endpoint in X.

Ptolemaic Graphs. A graph G is a *Ptolemaic graph* if for every four vertices u, v, w, x in the same connected component, $d(u,v)d(w,x) + d(u,x)d(v,w) \geqslant d(u,w)d(v,x)$.

▶ **Proposition 2** (Theorems 2.1,3.2 [15]). *Given a graph G, the following statements are equivalent: i) G is Ptolemaic, ii) G is both chordal and distance hereditary, and iii) G does not contain a gem or a chordless cycle as an induced subgraph.*

We call gems and chordless cycles *obstructions* and say that G contains an obstruction, if it has an obstruction as an induced subgraph.

Inter-Clique Digraphs. For a graph G, let $\mathcal{C}(G)$ be the set that contains i) all maximal cliques in G, and ii) non-empty intersections of (any number of) distinct maximal cliques in G. Consider the directed graph D_G which has a vertex v_C for each $C \in \mathcal{C}(G)$. For every $X, Y \in \mathcal{C}(G)$, where i) $X \subset Y$, and ii) there is no $W \in \mathcal{C}(G)$ such that $X \subset W$ and $W \subset Y$; we add an arc from v_Y directed towards v_X. For a directed graph T, $Und(T)$ denotes the underlying undirected graph (obtained by removing the directions associated with arcs in T). We make use of a characterization of Ptolemaic Graphs based on Inter-Clique Digraphs by Uehara and Uno[21].

▶ **Proposition 3** (Theorem 5,8 [21]). *A graph G is Ptolemaic if and only if $Und(D_G)$ is a forest. Moreover, the inter-clique digraph of a Ptolemaic graph can be computed in linear time.*

For a Ptolemaic graph G, let T_G denote its inter-clique digraph. We refer to the vertices in T_G (and $Und(T_G)$) as *bags*, to avoid confusions. A *leaf* of T_G is a leaf in $Und(T_G)$. For a bag $B \in T_G$, denote the associated set of vertices in G in the bag B by $V(B)$.

In Weighted Ptolemaic Deletion, we are given a graph G and a weight function $w : V \to \mathbb{R}^+ \cup \{0\}$, and the objective is to compute a minimum weight subset $X \subseteq V(G)$ such that $G \setminus X$ is a Ptolemaic graph.

For a graph G, we say that a set $X \subseteq V(G)$ is a *solution* for G if $G \setminus X$ is a Ptolemaic graph. By $\mathsf{opt}(G)$, we denote the size of a minimum sized solution for G. Moreover, if we are given a weight function $w : V(G) \to \mathbb{R}^+ \cup \{0\}$, then $\mathsf{opt}(G)$ denotes the weight of a minimum weight solution for G.[1] The following result is known regarding Weighted Ptolemaic Deletion.

▶ **Proposition 4** (Theorem 1.1 [7]). *There is a constant $c \in \mathbb{N}$ and a polynomial time algorithm* Approx *for* Weighted Ptolemaic Deletion*, which given a graph G and a weight function $w : V(G) \to \mathbb{R}^+ \cup \{0\}$, outputs a solution for G of weight at most $c \cdot \mathsf{opt}(G)$.*

2.1 Computing a Redundant Solution

Our kernelization algorithm will also use a "strengthened" approximate solution, which will be slightly larger in size than the solution that we obtain using the known constant-factor

[1] For a subset $S \subseteq V(G)$, $w(S) = \sum_{s \in S} w(S)$.

approximation algorithm (Proposition 4). The type of strengthening of an approximate solution that we use will be the same as the notion of *redundant solution* that was introduced in the context of INTERVAL VERTEX DELETION [4]. Intuitively speaking, a redundant solution allows us to "forget" about "small" obstructions in some of our reduction rules. This is achieved with the help of an implicit family of subsets of $V(G)$ which has a guarantee that any solution for G of size at most k is a hitting set for this family. We will now formalise the above notions.

Consider a graph G. For a family $\mathcal{W} \subseteq 2^{V(G)}$, a subset $S \subseteq V(G)$ *hits* $\mathcal{W}$ if for all $W \in \mathcal{W}$, we have $S \cap W \neq \emptyset$. We say that $\mathcal{W} \subseteq 2^{V(G)}$ is *t-necessary* if every solution for G of size at most t hits $\mathcal{W}$. Moreover, we say that an obstruction $\mathbb{O}$ is *covered by* $\mathcal{W}$ if there exists $W \in \mathcal{W}$, such that $W \subseteq V(\mathbb{O})$. Now, we are ready to formally define the notion of redundant solution (for our context).

▶ **Definition 5.** For a graph G, a family $\mathcal{W} \subseteq 2^{V(G)}$ and $t \in \mathbb{N}$, a subset $M \subseteq V(G)$ is *t-redundant with respect to* $\mathcal{W}$ if for any obstruction $\mathbb{O}$ that is not covered by $\mathcal{W}$, it holds that $|M \cap V(\mathbb{O})| > t$.

Let (G, k) be an instance of PTOLEMAIC DELETION. We will next focus on computing a t-redundant solution for G, for an appropriately defined $\mathcal{W}$ (and t). The crux of the algorithm for computing a redundant solution for INTERVAL VERTEX DELETION as described in [4] is to make a set of vertices "undeletable". It then computes an approximate solution that does not contain any of these vertices. If such an approximate solution is too large, then the set of vertices is added as a member of the necessary family. Otherwise, the redundancy of the solution is enhanced. In our case, we use the approximation algorithm for the weighted version of PTOLEMAIC DELETION to determine this, by setting the weights of the "undeletable" vertices very large. Our objective for the remainder of the subsection will be to prove the following lemma.

▶ **Lemma 6.** *In polynomial time we can either correctly conclude that (G, k) is a* NO*-instance, or compute a $(k+1)$-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ and a set $M \subseteq V(G)$, such that $\mathcal{W} \subseteq 2^M$, M is a solution for G that is 4-redundant with respect to $\mathcal{W}$ and $|M| \in O(k^5)$.*

For an integer ℓ and set $U \subseteq V(G)$, we let $w^*_{\ell,U} : V(G) \to \mathbb{R}_{\geqslant 0}$ be the function such that, for each $v \in V(G) \setminus U$, $w^*_{\ell,U}(v) = 1$ and for each $v \in U$, $w^*_{\ell,U}(v) = c\ell + 1$. We show the following results towards proving Lemma 6.

▶ **Lemma 7.** *Consider a set $U \subseteq V(G)$ and an integer $\ell \in \mathbb{N}$. For the* WEIGHTED PTOLEMAIC DELETION *instance $(G, w^*_{\ell,U})$, if* Approx *returns a solution of weight more than $c\ell$, then the family $\{U\}$ is ℓ-necessary.*[2]

Proof. Suppose that A is the output of Approx for (G, w^*), such that $w^*(A) > c\ell$. From the above we can obtain that $\mathsf{opt}(G, w) > \ell$. Thus we can conclude that any solution for G of size at most ℓ must contain a vertex from U, and hence $\{U\}$ is ℓ-necessary. ◀

▶ **Lemma 8.** *Consider $U \subseteq V(G)$ and $\ell \in \mathbb{N}$. For the* WEIGHTED PTOLEMAIC DELETION *instance $(G, w^*_{\ell,U})$, if* Approx *returns a solution A with $w^*_{\ell,U}(A) \leqslant c\ell$, then for every obstruction $\mathbb{O}$ in G, $|V(\mathbb{O}) \cap U| + 1 \leqslant |V(\mathbb{O}) \cap (U \cup A)|$.*

Proof. As $w^*_{\ell,U}(A) \leqslant c\ell$, we obtain that $A \cap U = \emptyset$. Since A is a solution for G, any obstruction in it must contain at least one vertex from A. Hence the result follows. ◀

---

[2] The constant c and Approx are the constant and the algorithm, respectively, from Proposition 4.

We are now ready to prove Lemma 6.

Proof of Lemma 6. We design an algorithm, RedundantPD, to compute a redundant solution as required by the lemma. Given an instance (G, k) of Ptolemaic Deletion, we do the following. Let M_0 to be the output of Approx, for the Weighted Ptolemaic Deletion instance $(G, w^*_{k+1,\emptyset})$ (see Proposition 4). Let $\mathcal{W}_0 := \emptyset$ and $\mathcal{T}_0 := \{(v) \mid v \in M_0\}$. If $|M_0| > c(k+1)$, we can correctly conclude that (G, k) is a no-instance. Otherwise, for $i = 1, 2, 3, 4$ (in this order), the algorithm executes the following steps:

1. Initialize $M_i := M_{i-1}$, $\mathcal{W}_i := \mathcal{W}_{i-1}$ and $\mathcal{T}_i := \emptyset$.
2. For every tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$:
   a. Let A be the output of Approx for the Weighted Ptolemaic Deletion instance $(G, w^*_{k+1,\{v_0,v_1 \ldots v_{i-1}\}})$.
   b. If $w(A) > c(k+1)$, then add $\{v_0, v_1, \ldots, v_{i-1}\}$ to $\mathcal{W}_i$.
   c. Otherwise, insert every vertex in $A \setminus \{v_0, v_1, \ldots, v_{i-1}\}$ into M_i, and for all $u \in A \setminus \{v_0, v_1, \ldots, v_{i-1}\}$, insert $(v_0, v_1, \ldots, v_{i-1}, u)$ into $\mathcal{T}_i$.

Equipped with Lemma 7 and 8, we can show that RedundantPD satisfies the required conditions of the lemma, using arguments identical to the proof of Lemma 3.1 in [4]. ◀

The utility of a redundant solution is captured in the following lemma.

▶ **Lemma 9.** *Consider an instance (G, k) of* Ptolemaic Deletion*, and D be a 4-redundant solution for G with respect to a $(k+1)$-necessary family $\mathcal{W}$, returned by Lemma 6. For $v \in V(G) \setminus D$, let X be any solution of size at most k (if it exists) for $G \setminus \{v\}$. Then X hits all obstructions of size at most 5 in G. In other words, $G \setminus X$ does not contain a gem or a chordless cycle of length less than 6.*

Proof. Notice that $X \cup \{v\}$ is a solution for G, of size at most $k+1$. Any obstruction $\mathbb{O}$ in $G \setminus X$ must contain the vertex v. Consider such an obstruction $\mathbb{O}$. If $\mathbb{O}$ is covered by $\mathcal{W}$, then there must exist a $W \in \mathcal{W}$ so that $V(\mathbb{O}) \subseteq W$. Since $\mathcal{W}$ is $(k+1)$-necessary, $\mathcal{W} \subseteq 2^D$, and $v \in V(G) \setminus D$, we can obtain that $X \cap W \neq \emptyset$, and therefore $G \setminus X$ does not contain $\mathbb{O}$. Otherwise, $\mathbb{O}$ is not covered by $\mathcal{W}$, and thus, $|V(\mathbb{O}) \cap D| \geqslant 5$. Since $v \in V(\mathbb{O}) \setminus D$, it follows that $V(\mathbb{O}) \geqslant 6$. ◀

3 Kernel for Ptolemaic Vertex Deletion

The objective of this section is to prove Theorem 1. Let (G, k) be an instance of Ptolemaic Deletion. Our algorithm begins by computing a c-approximate solution S, for the Weighted Ptolemaic Deletion instance $(G, w^*_{n+1,\emptyset})$ using Approx (see Proposition 4).[3] If $|S| > ck$, we conclude that (G, k) is a no-instance, and return a trivial no-instance of the problem as a kernel. We now assume $|S| \leqslant ck$. Next, we use Lemma 6 to compute (in polynomial time) a $(k+1)$-necessary family $\mathcal{W}$ and a 4-redundant (with respect to $\mathcal{W}$) solution D for G of size bounded by $\mathcal{O}(k^5)$. Note that we have $\mathcal{W} \subseteq 2^D$, from the lemma.

Firstly, we bound the size of a maximal clique in $G \setminus (S \cup D)$ using the structure of the connected components upon removal of a maximal clique in a Ptolemaic graph and the properties guaranteed by the 4-redundant solution D. Intuitively, for each maximal clique C in $G \setminus S$, we do the following. Firstly we mark a few vertices in C which are neighbours of some vertices in S. Secondly, we create auxiliary graphs, with the guarantee that: i) if

[3] The choice of $n+1$ is arbitrary.

the maximum matching in these auxiliary graphs is "large", we find a vertex v that can be safely deleted to obtain the instance $(G \setminus \{v\}, k-1)$, and ii) otherwise, we will be able to use maximum matchings in these auxiliary graphs to mark at most $\mathcal{O}(k^3)$ many vertices in C. The marked neighbours of S in C and the vertices marked using the help of auxiliary graphs will help us capture different behaviours of C in an obstruction. Finally we argue that deleting unmarked vertices in $C \setminus D$ is safe. Formally, we prove the following lemma in Section 4.

▶ **Lemma 10.** *In polynomial time we can either conclude that (G, k) is a* NO*-instance, or obtain an equivalent instance (G', k'), such that $k' \leqslant k$, G' is an induced subgraph of G, such that $S \cup D \subseteq V(G')$, and each (maximal) clique in $G' \setminus (S \cup D)$ has at most $\mathcal{O}(k^3)$ vertices.*

If at any point in our algorithm, we are able to conclude that the given instance is a NO-instance, then we output some trivial NO-instance as a kernel with $\mathcal{O}(1)$ vertices. We use Lemma 10 for the given instance, and without loss of generality assume that we obtained an equivalent instance of Ptolemaic Deletion, which satisfies all the conditions guaranteed by the lemma. For the sake of notational simplicity, we use (G, k) as the instance of Ptolemaic Deletion that we currently have, i.e., the one returned by the lemma.

Next, we use the fact that the undirected version of the inter-clique digraph of a Ptolemaic graph is an undirected forest. We construct a suitable Ptolemaic graph $G \setminus (S \cup R)$ for a carefully constructed set $R \subseteq D$ and reduce the number of leaves in its undirected inter-clique digraph. Towards this, we use the idea of bounding the independence degree of vertices from [3]. We introduce two sets of pairs of vertices E_M and E_I, which will stand for mandatory and irrelevant pairs of vertices. Moreover, we call an edge $\{u, v\} \in E(G)$ *mandatory* if $\{u, v\} \in E_M$ and *irrelevant* if $\{u, v\} \in E_I$. An edge is *relevant* if it is not irrelevant.

For a graph H and sets $E_M, E_I \subseteq V(H) \times V(H)$, for a vertex $v \in V(H)$, by $N^H_{\mathsf{rel}}(v)$ we denote the set of vertices in the neighbourhood of v in H which are adjacent to v using an edge in $E(H) \setminus E_I$. We omit the superscript H from the above notation, whenever the context is clear.

▶ **Definition 11.** *Given a set $E_I \subseteq V(G) \times V(G)$ and a vertex $v \in V(G)$, the independence degree of v in G, denoted by $d^G_I(v)$, is the size of a maximum independent set in the graph $G[N_{\mathsf{rel}}(v)]$.*

We remark that we omit the superscript in $d^G_I(v)$ when the graph is clear from the context. Next, we consider an annotated version of Ptolemaic Deletion similar to [3].

Augmented Ptolemaic Deletion	**Parameter:** k
Input: An undirected graph G, sets $E_M, E_I \subseteq V(G) \times V(G)$ and a non-negative integer k, such that any $X \subseteq V(G)$ which hits all chordless cycles in G which contain no edge from E_I, hits all chordless cycles in G.	
Question: Does there exist a subset $X' \subseteq V(G)$ (called a *solution* for (G, k, E_M, E_I)), such that (i) $\lvert X' \rvert \leqslant k$ (ii) For each $\{u, v\} \in E_M$, $X' \cap \{u, v\} \neq \emptyset$ and (iii) $G \setminus X'$ is a Ptolemaic graph?	

We next obtain an instance of Augmented Ptolemaic Deletion, where the number of leaves in the inter-clique digraph of the Ptolemaic graph obtained after removing S and a suitable subset of D is bounded by $\mathcal{O}(k^3)$.

▶ **Lemma 12.** *In polynomial time, either we can conclude that (G, k) is a* NO*-instance, or output an instance (G', k', E'_M, E'_I) of* Augmented Ptolemaic Deletion*, a set $R \subseteq D$ and the inter-clique digraph $T_{G' \setminus (R \cup S)}$ of $G' \setminus (R \cup S)$ so that the following hold:*

1. *$k' \leqslant k$, $S, D \subseteq V(G')$, and G' is an induced subgraph of G.*
2. *for each $\{u,v\} \in E'_M$, either $u \in S$ or $v \in S$.*
3. *Each chordless cycle in G' which contains a vertex from R contains an edge from E'_I.*
4. *(G,k) is a* YES*-instance of* Ptolemaic Deletion *if and only if (G', k', E'_M, E'_I) is a* YES*-instance of* Augmented Ptolemaic Deletion*.*
5. *The neighbourhood of every vertex $r \in R$ in $G \setminus (R \cup S)$ is a clique.*
6. *The number of leaves in $T_{G' \setminus (R \cup S)}$ is bounded by $\mathcal{O}(k^3)$.*
7. *The independence degree (in G') of every vertex $s \in S$ is bounded by $\mathcal{O}(k^2)$ and $|E'_M| \in \mathcal{O}(k^2)$.*

The proof of the above lemma can be found in Section 5. We use Lemma 12, and assume that we have an instance (G', k', E'_M, E'_I) of Augmented Ptolemaic Deletion and a set $R \subseteq D$, satisfying the properties guaranteed by the lemma. Using the next lemma we obtain an instance of Augmented Ptolemaic Deletion equivalent to that of (G', k', E'_M, E'_I), with $\mathcal{O}(k^6)$ vertices.

▶ **Lemma 13.** *For the* Ptolemaic Deletion *instance (G,k), let (G', k', E'_M, E'_I) be the instance of* Augmented Ptolemaic Deletion *and $R \subseteq D$ be the set returned by Lemma 12.*[4] *In polynomial time we either conclude that (G', k', E'_M, E'_I) is a* NO*-instance or output an equivalent instance $(\widetilde{G}, \widetilde{k}, \widetilde{E}_M, \widetilde{E}_I)$ of* Augmented Ptolemaic Deletion*, such that $\widetilde{k} \leqslant k'$, $|V(\widetilde{G})| \in \mathcal{O}(k^6)$, and $|\widetilde{E}_M| \leqslant |E'_M|$.*

The above lemma is formally proved in Section 6. A rough sketch on how this lemma is proved as follows. In Section 6 we reduce the number of vertices in 2-degree bags in the undirected graph corresponding to the inter-clique digraph of $G \setminus (R \cup S)$. Since we have a bound on the number of leaves from Lemma 12, we get a bound on the number of bags of degree 3 or more as well. Thus, proving a bound on the number of vertices present in bags of degree 2 gives us a bound on the total number of vertices in the inter-clique digraph of $G \setminus (R \cup S)$, since we already have a bound on clique size (and hence, on the number of vertices in any bag) from Lemma 10. Accounting for each step we obtain an Augmented Ptolemaic Deletion instance with $\mathcal{O}(k^6)$ vertices.

Equipped with Lemmas 10 to 13, we are now ready to prove Theorem 1.

Proof of Theorem 1. Consider an instance $(\widehat{G}, \widehat{k})$ of Ptolemaic Deletion. If Lemma 10 returns that $(\widehat{G}, \widehat{k})$ is a NO-instance of Ptolemaic Deletion, then we return some trivial NO-instance with $\mathcal{O}(1)$ vertices as a kernel for Ptolemaic Deletion. Otherwise, we have an equivalent instance (G,k), where each (maximal) clique in $G \setminus (S \cup D)$ has at most $\mathcal{O}(k^3)$ vertices. Next we apply Lemma 12 for the instance (G,k). Again, if the lemma returns that (G,k) is a NO-instance, we return a trivial $\mathcal{O}(1)$-vertex kernel. Otherwise, we have an Augmented Ptolemaic Deletion instance (G', k', E'_M, E'_I) and the set $R \subseteq D$, returned by the lemma. We next apply Lemma 13. Firstly consider the case when the lemma returns that (G', k', E'_M, E'_I) is a NO-instance of Augmented Ptolemaic Deletion. Then together with Lemma 12, we have that (G,k), and hence, $(\widehat{G}, \widehat{k})$ is a NO-instance of Ptolemaic Deletion. Now consider the case when Lemma 13 returns an instance $(\widetilde{G}, \widetilde{k}, \widetilde{E}_M, \widetilde{E}_I)$ of Augmented Ptolemaic Deletion, where $|V(\widetilde{G})| \in \mathcal{O}(k^6)$. From Lemmas 10 to 13, we obtain that $(\widehat{G}, \widehat{k})$ is a YES-instance of Ptolemaic Deletion if and only if $(\widetilde{G}, \widetilde{k}, \widetilde{E}_M, \widetilde{E}_I)$ is a YES-instance of Augmented Ptolemaic Deletion.

[4] We assume that in G, each maximal clique has at most $\mathcal{O}(k^3)$ vertices, using Lemma 10.

We next obtain an instance (G', k') of Ptolemaic Deletion such that $|V(G')| \in \mathcal{O}(k^6)$ and (G', k') is a yes-instance of Ptolemaic Deletion if and only if $(\widetilde{G}, \widetilde{k}, \widetilde{E}_M, \widetilde{E}_I)$ is a yes-instance of Augmented Ptolemaic Deletion. Initialize $G' = \widetilde{G}$ and set $k' = \widetilde{k}$. For each $\{u, v\} \in \widetilde{E}_M$, add $k + 2$ vertex disjoint paths between u and v using 2 new vertices for each path. As $|\widetilde{E}_M| \leqslant |E'_M| \in \mathcal{O}(k^2)$, we obtain that $|V(G')| \in \mathcal{O}(k^6)$. Notice that no gem in G' can contain a vertex from $V(G') \setminus V(\widetilde{G})$. For every $\{u, v\} \in \widetilde{E}_M$ and any solution X to the Ptolemaic Deletion instance (G', k') , we must have either $u \in X$ or $v \in X$, for if not, then $G' \setminus X$ must contains a chordless cycle through u, v by construction. It follows that (G', k') is a yes-instance of Ptolemaic Deletion if and only if $(\widetilde{G}, \widetilde{k}, \widetilde{E}_M, \widetilde{E}_I)$ is a yes-instance of Augmented Ptolemaic Deletion. Hence, it must be the case that (G', k') and $(\widehat{G}, \widehat{k})$ are equivalent instances of Ptolemaic Deletion, where $|V(G')| \in \mathcal{O}(k^6)$ and $k' \leqslant k$. This concludes the proof. ◀

4 Bounding Sizes of Maximal Cliques in $G \setminus (S \cup D)$

The objective of this section is to prove Lemma 10. As S is a solution for G, $G \setminus S$ does not have any chordless cycles. We next state a well-known result regarding enumerating maximal cliques in chordal graphs.

▶ **Proposition 14** (see, for example, Theorem 4.8 [14]). *In polynomial time we can compute the set of all maximal cliques in a given chordal graph.*

From the above proposition, the number of distinct cliques and the time required to enumerate them, are both bounded by $n^{\mathcal{O}(1)}$. Thus, for the remainder of this section, we work with a fixed maximal clique C in $G \setminus S$, and our objective will be to either conclude that the number of vertices in $C \setminus D$ is bounded by $\mathcal{O}(k^3)$, or find a vertex in C which we can safely delete from G. To achieve this, we will design some marking schemes, and show that if there are unmarked vertices, then we can safely delete them. We start with a useful lemma regarding the structure of connected components upon removal of a maximal clique from a Ptolemaic graph.

▶ **Lemma 15.** *Let $\widehat{G}$ be a Ptolemaic graph and C be a maximal clique in $\widehat{G}$. Let A be a connected component of $\widehat{G} \setminus C$. Then for every vertex $v \in V(A)$ we must have $N_{\widehat{G}}(v) \cap C = \emptyset$ or $N_{\widehat{G}}(v) \cap C = N_{\widehat{G}}(V(A)) \cap C$.*

Proof. Consider a connected component A of $\widehat{G} \setminus C$. Towards a contradiction suppose that the result is not true, and let $u, v \in V(A)$ be two distinct vertices such that $N_{\widehat{G}}(u) \cap C, N_{\widehat{G}}(v) \cap C \neq \emptyset$ and $N_{\widehat{G}}(u) \cap C \neq N_{\widehat{G}}(v) \cap C$. Furthermore, let u and v be a pair of vertices satisfying the above property that have shortest possible distance between them in A. Note that $C \not\subseteq N_{\widehat{G}}(u)$ and $C \not\subseteq N_{\widehat{G}}(v)$, as C is a maximal clique in $\widehat{G}$.

Suppose that $\{u, v\} \in E(\widehat{G})$. If there exist vertices $a, b \in V(C)$ such that $a \in N_{\widehat{G}}(u) \setminus N_{\widehat{G}}(v)$, $b \in N_{\widehat{G}}(v) \setminus N_{\widehat{G}}(u)$ then $\widehat{G}[\{u, v, a, b\}]$ is a chordless cycle in $\widehat{G}$, which is a contradiction, as $\widehat{G}$ is Ptolemaic graph. Therefore, either $N_{\widehat{G}}(u) \cap C \subseteq N_{\widehat{G}}(v) \cap C$ or $N_{\widehat{G}}(v) \cap C \subseteq N_{\widehat{G}}(u) \cap C$. Assume without loss of generality that $N_{\widehat{G}}(u) \cap C \subseteq N_{\widehat{G}}(v) \cap C$. This together with the choice of u and v implies that there are vertices $a \in (N_{\widehat{G}}(v) \cap C) \setminus N_{\widehat{G}}(u)$, $b \in N_{\widehat{G}}(u) \cap C$ and $x \in C \setminus (N_{\widehat{G}}(u) \cup N_{\widehat{G}}(v))$. But then, $\widehat{G}[\{a, b, x, u, v\}]$ is a gem in $\widehat{G}$, which is a contradiction.

Now suppose $\{u, v\} \notin E(\widehat{G})$ and consider the shortest path P between u, v in A. Note that P has at least one more vertex apart from u and v, as $\{u, v\} \notin E(\widehat{G})$. Firstly consider the case when there are vertices $a \in (N_{\widehat{G}}(u) \cap C) \setminus N_{\widehat{G}}(v)$, $b \in (N_{\widehat{G}}(v) \cap C) \setminus N_{\widehat{G}}(u)$. Note that by the choice of u and v (of them being at shortest distance in A satisfying the assumed properties), no internal vertex of P can be adjacent to a or b. But then P along with the edges $\{a,b\},\{a,u\}$ and $\{v,b\}$ forms a chordless cycle in $\widehat{G}$, which is a contradiction. We now consider the case when there are no such a and b, as assumed previously. This implies that either $N_{\widehat{G}}(u)\cap C \subseteq N_{\widehat{G}}(v)\cap C$ or $N_{\widehat{G}}(v)\cap C \subseteq N_{\widehat{G}}(u)\cap C$. Suppose that $N_{\widehat{G}}(u)\cap C \subseteq N_{\widehat{G}}(v)\cap C$ (the other case is symmetric). Now consider a vertex $a \in N_{\widehat{G}}(v)\cap N_{\widehat{G}}(u)\cap C$. If there is an internal vertex of P that is not adjacent to a, then we conclude that $\widehat{G}[V(P)\cup\{a\}]$ contains a chordless cycle. If every internal vertex of P is adjacent to a, P must have exactly 1 vertex, say w, apart from $\{u,v\}$, since otherwise we would obtain a gem obstruction in $\widehat{G}[V(P)\cup\{a\}]$. Since $N_{\widehat{G}}(u)\cap C \neq N_{\widehat{G}}(v)\cap C$ we must have either $N_{\widehat{G}}(w)\cap C \neq N_{\widehat{G}}(u)\cap C$ or $N_{\widehat{G}}(w)\cap C \neq N_{\widehat{G}}(v)\cap C$. In either case, since $\{u,w\},\{w,v\} \in E(\widehat{G})$, this is impossible from the analysis in the preceeding paragraph of the case when $\{u,v\} \in E(\widehat{G})$. It follows that such an a cannot exist, and therefore $N_{\widehat{G}}(v)\cap N_{\widehat{G}}(u)\cap C = \emptyset$. Together, we must have either $N_{\widehat{G}}(v)\cap C = \emptyset$ or $N_{\widehat{G}}(u)\cap C=\emptyset$, a contradiction.

◀

Recall that we computed D, a 4-redundant solution with respect to $\mathcal{W}$. This will allow us to only focus on chordless cycles of length at least 6 (see Lemma 9). A chordless cycle can contain at most 2 vertices from C, as C is a (maximal) clique. If C has many vertices, we would like to argue that, we can find a vertex $c \in C \setminus D$ to delete from G. Roughly speaking, to achieve this, for every pair of vertices u, v, that are neighbours of c in some chordless cycle, we would like to ensure that, after removing some solution of size at most k, there is either (i) at least one marked common neighbour $c' \in C$ of u and v or (ii) two distinct marked vertices $c_1, c_2 \in C$, such that $\{u,c_1\},\{v,c_2\} \in E(G)$. We show, through a simple observation, that preserving such marked vertices is enough to safely delete the vertex c from the input instance (G,k) to obtain the instance $(G\setminus\{c\},k)$.

Towards formally defining our reduction rules and marking schemes, we begin by giving some useful results and fixing some notations. We start with a simple observation that will be useful while deleting vertices from C.

▶ **Observation 16.** *Consider a graph $\widehat{G}$ and a spanning cycle K of $\widehat{G}$. Then the graph $\widehat{G}$ contains an obstruction if one of the following holds: i) $|V(\widehat{G})| \geqslant 5$ and there is $v \in V(K)$, such that for each $\{x,y\} \in E(\widehat{G})\setminus E(K)$, $v \in \{x,y\}$, or ii) $|V(\widehat{G})| \geqslant 7$ and there is $\{u,v\} \in E(K)$, such that for each $\{x,y\} \in E(\widehat{G})\setminus E(K)$, we have $\{u,v\}\cap\{x,y\} \neq \emptyset$.*

Proof. To prove (i), notice that if v is not adjacent (in $\widehat{G}$) to at least one vertex in $V(K)\setminus\{v\}$, then $\widehat{G}$ must contain a chordless cycle. Otherwise, since K has at least 5 vertices, any four consecutive vertices a,b,c,d on K, all different from v, together with v give us a gem.

To prove (ii), let us consider five consecutive vertices of the cycle K starting from u, v, taken in order, say u,v,a,b,c. We claim that apart from its two neighbours u and a in K, the vertex v can be adjacent to only the vertices b and c in $\widehat{G}$. Suppose this is not the case. Then we get a cycle K' with $V(K') \subseteq V(K)$ in $\widehat{G}$ of length at least 5 which does not include u. Applying part (i) with the graph $\widehat{G}[V(K')]$ then gives an obstruction in $\widehat{G}$.

Therefore v can only be adjacent to u,a,b,c in K. Consider the case when v is adjacent to b but not c. Let K' be the cycle in $\widehat{G}$, obtained from K by deleting a and adding the edge $\{v,b\}$. Notice that $\widehat{G}[V(K')]$ and the spanning cycle K' of $\widehat{G}[V(K')]$, satisfy part (i) of the observation. Thus, $\widehat{G}[V(K')]$ (and hence $\widehat{G}$) must contain an obstruction. For the case when v is adjacent to c, we can obtain an obstruction in $\widehat{G}$ by using arguments similar to our previous case. Finally, if v is not adjacent to both b and c, then $\widehat{G}$ and K satisfy the conditions of (i). ◀

Now we define the notions of replacement vertices.

▶ **Definition 17.** Given a vertex $c \in C$, a set $X \subseteq V(G) \setminus \{c\}$ of size at most k, and a chordless cycle K of length at least 6 in $G \setminus X$ containing c, a vertex $c' \in C \setminus (X \cup V(K))$ is a *replacement* for c in K, if c' is adjacent to both the neighbours of c in K.

▶ **Definition 18.** Given a vertex $c \in C$, a set $X \subseteq V(G) \setminus \{c\}$ of size at most k, and a chordless cycle K of length at least 6 in $G \setminus X$ containing c, a pair of distinct vertices $c_1, c_2 \in C \setminus (X \cup V(K))$, is a *replacement* for c in K, if $\{u, c_1\}, \{v, c_2\} \in E(G)$, where u, v are the neighbours of c in K.

The purpose of this notion of replacements for a vertex is captured in the following lemma whose proof follows from Observation 16.

▶ **Lemma 19.** *Consider any vertex $c \in C$. Let $X \subseteq V(G)$ be a set of vertices such that $c \notin X$. Suppose that $G \setminus X$ contains a chordless cycle K of length at least 6 which includes the vertex c. Now suppose that there is either a single replacement vertex c' or a pair of replacement vertices c_1, c_2 for c in K, such that $c' \in C \setminus (X \cup V(K))$ and $c_1, c_2 \in C \setminus (X \cup V(K))$, as the case may be. Then $G \setminus (X \cup \{c\})$ must contain an obstruction.*

Proof. Let u and v be the neighbours of c in K. If we have a single replacement vertex c' for c in K, then consider the cycle K' obtained by replacing the path ucv by the path $uc'v$ in K. The cycle K' is clearly of length at least 6. Applying part (i) of Observation 16 to the graph $G[V(K')]$, we must have an obstruction in $G \setminus (X \cup \{c\})$. Likewise, if we have a pair of distinct vertices $c_1, c_2 \in C$ as a replacement for c in K, then consider the cycle K' obtained by replacing the path ucv by the path uc_1c_2v. The cycle K' is of length at least 7, therefore by item (ii) of Observation 16 applied to the graph $G[V(K')]$, $G \setminus (X \cup \{c\})$ must contain an obstruction.

◀

Now suppose that we mark a certain number of vertices in C. Consider any unmarked vertex $c \in C \setminus D$, and let X be a solution of size at most k for $G \setminus \{c\}$. If X is not a solution for G, then $G \setminus X$ must contain an obstruction that contains the vertex c. This obstruction must be a chordless cycle K of length at least 6 since $c \notin D$. If we design our marking scheme in such a way that for every such obstruction K, there is either (i) a marked replacement vertex c' such that $c' \notin (X \cup V(K))$ or (ii) a pair of distinct marked replacement vertices c_1, c_2 for c in K such that $c_1, c_2 \notin (X \cup V(K))$ as per the definitions above, then by Lemma 19, $G \setminus (X \cup \{c\})$ must contain an obstruction - a contradiction since X was a solution for $G \setminus \{c\}$. It follows that X must be a solution for G, and therefore the instances (G, k) and $(G \setminus \{c\}, k)$ are equivalent. Thus, given that we design our marking scheme to satisfy the above, we may delete an unmarked vertex from the input instance G without reducing the parameter.

We now describe our marking scheme and formalize the ideas from above, by looking at where the neighbours of an unmarked vertex in a chordless cycle can potentially lie. Whenever we say - mark any ℓ vertices satisfying a property P, if there are less than ℓ vertices satisfying P, then we mark all vertices which satisfy P. We start by designing our marking scheme to handle the case when the neighbours of an unmarked vertex in a chordless cycle lie in S.

▶ **Marking Scheme 1.** *For every pair of (not necessarily distinct) vertices $s_1, s_2 \in S$, mark any $k + 2$ vertices in $N(s_1) \cap N(s_2) \cap C$.*

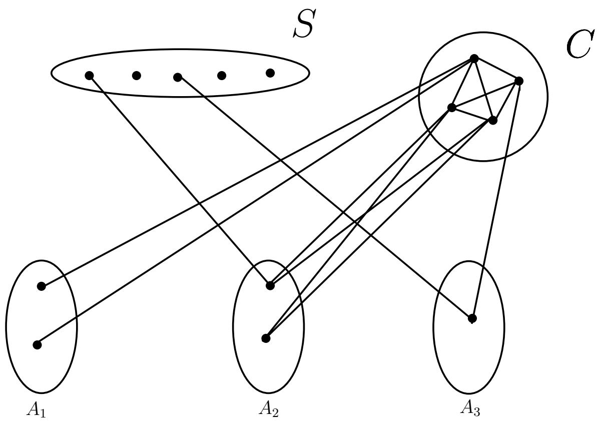

Figure 1 Bounding the size of a maximal clique C. The figure shows the components of $G \setminus (S \cup C)$ for a particular instance. For every component A and vertex $v_A \in A$, $N(v_A) \cap C = \emptyset$ or $N(v_A) \cap C = N(A) \cap C$.

To consider the cases when the neighbours of a potential unmarked vertex lie in connected components of $G \setminus (S \cup C)$, we make use of two auxiliary bipartite graphs H_s^1 and H_s^2, corresponding to each vertex s in S. Let us denote the set of connected components of $G \setminus (S \cup C)$ by $\mathcal{A}$(see Figure 1). The vertex set of H_s^1 is $V_s^1 \cup V_2$ where $V_s^1 = \{c \in C \ : (s, c) \notin E(G)\}$ and $V_2 = \{v_A \ : A \in \mathcal{A}\}$. We add an edge from a vertex $c \in V_s^1$ and a vertex $v_A \in V_2$ whenever both c and s have (possibly different) neighbours in A. The vertex set of H_s^2 is $V_s^2 \cup V_2$, where $V_s^2 = \{c \in C \ : (s, c) \in E(G)\}$. We add an edge from a vertex $c \in V_s^2$ and a vertex $v_A \in V_2$ whenever there is an obstruction in the subgraph induced on $\{s, c\} \cup V(A)$. We have the following observation that follows from the above description and Proposition 3.[5]

▶ **Observation 20.** *For $s \in S$, H_s^1 and H_s^2 can be constructed in polynomial time.*

Roughly speaking, we next show that, for $s \in S$ and two edges of H_s^1, incident to different vertices in V_2, we can construct an obstruction in G.

▶ **Lemma 21.** *Consider $s \in S$ and edges $\{c_1, v_{A_1}\}, \{c_2, v_{A_2}\} \in E(H_s^1)$, where $c_1, c_2 \in V_s^1$ and $A_1, A_2 \in \mathcal{A}$, where $A_1 \neq A_2$. Then, $G[V(A_1) \cup V(A_2) \cup \{s, c_1, c_2\}]$ has a chordless cycle.*

Proof. By the construction of V_s^1, $\{s, c_1\}, \{s, c_2\} \notin E(G)$. For $i, j \in \{1, 2\}$, let P_{ij} (if it exists) be a shortest path between s and c_i such that every internal vertex is from A_j. By the construction of H_s^1, P_{11} and P_{22} must necessarily exist. Notice that if at least one of c_1 or c_2, say c_1, is a neighbour of both v_{A_1} and v_{A_2} in H_s^1, then $G[\{s, c_1\} \cup V(P_{11}) \cup V(P_{12})]$ is a chordless cycle. Otherwise, the vertices s, c_1, c_2 together with the paths P_{11}, P_{22} and the edge $\{c_1, c_2\}$ constitute a chordless cycle $G[\{s, c_1, c_2\} \cup V(P_{11}) \cup V(P_{22})]$ in $G[V(A_1) \cup V(A_2) \cup \{s, c_1, c_2\}]$. ◀

For $s \in S$, we now attempt to find $(k+2)$-sized matchings in the graphs H_s^1 and H_s^2. Note that the existence of a $(k+2)$-sized matching in H_{s1} would imply that (at least) two

[5] We note that there are other well-known polynomial time recognition algorithm for Ptolemaic graphs, that can also be used to infer that H_s^2, for $s \in S$, can be constructed in polynomial time.

matching edges "remain", even after we delete at most k vertices from G. The above together with Lemma 21 will allow us to conclude that s must belong to every solution for G of size at most k. The definition of H_s^2 implies that even a single "remaining" matching edge in H_s^2 would give us an obstruction, if s is not included in the solution.

▶ **Lemma 22.** *Consider $s \in S$, and let M_s^1 and M_s^2 be maximum matchings in H_s^1 and H_s^2, respectively. If either $|M_s^1| \geqslant k+2$ or $|M_s^2| \geqslant k+2$, then any solution for G of size at most k must include the vertex s.*

Proof. Towards a contradiction, suppose that X is a solution for G of size at most k which does not include the vertex s. Consider the case when $|M_s^1| \geqslant k+2$. It follows that there are distinct components $A_1, A_2 \in \mathcal{A}$, and vertices $c_1, c_2 \in V_s^1$, so that (i) $X \cap (V(A_1) \cup V(A_2) \cup \{c_1, c_2\}) = \emptyset$ and (ii) $\{c_1, v_{A_1}\}, \{c_2, v_{A_2}\} \in E(H_s^1)$ hold. But then Lemma 21 implies that $G \setminus X$ contains an obstruction, which is a contradiction. Next consider the case when $|M_s^2| \geqslant k+2$. Again as X is of size at most k, we can obtain that there is a component $A \in \mathcal{A}$ and a vertex $c \in V_s^2$ so that (i) $X \cap (V(A) \cup \{c\}) = \emptyset$ and (ii) $\{c, v_A\} \in E(H_s^2)$ hold. By the definition of H_s^2, this implies that $G \setminus X$ contains an obstruction in the subgraph induced on $V(A) \cup \{s, c\}$, which is a contradiction. ◀

Using the above lemma, we devise our first reduction rule.

▶ **Reduction Rule 1.** *If there is $s \in S$, such that $|M_s^1| \geqslant k+2$ or $|M_s^2| \geqslant k+2$, then return $(G \setminus \{s\}, k-1)$.*

The correctness of the above reduction rule follows from Lemma 22, and the polynomial time applicability of it follows from the fact that maximum matching in a graph can be computed in polynomial time (see, for example [12]). Hereafter we assume that Reduction rule 1 is not applicable, and we proceed with our next marking scheme.

▶ **Marking Scheme 2.** *For each $s \in S$, let M_s^1 and M_s^2 be maximum matchings in H_s^1 H_s^2, respectively. Then we do the following: i) for each $i \in \{1,2\}$, we mark all the vertices in $V(M_s^i) \cap C$, ii) for each $A \in \mathcal{A}$, where $v_A \in V(M_s^1)$, mark any $k+2$ vertices in $N(V(A)) \cap V_s^1$, and iii) for each $A \in \mathcal{A}$, where $v_A \in V(M_s^2)$, mark any $k+2$ vertices in $N(V(A)) \cap V_s^2$.*

The next reduction rule deletes unmarked vertices in C.

▶ **Reduction Rule 2.** *If there is $c \in C \setminus D$ that is not marked by Marking Scheme 1 or 2, then return $(G \setminus \{c\}, k)$.*

As Marking Scheme 1 and 2 can be executed in polynomial time, the above reduction rule can be executed in polynomial time as well. In the next lemma we show its safeness.

▶ **Lemma 23.** *Reduction rule 2 is safe.*

Proof. Consider $c \in C \setminus D$ that is not marked by Marking Scheme 1 or 2. As Ptolemaic graphs are closed under induced subgraphs, to prove the lemma, it is enough to argue that a solution X for $G \setminus \{c\}$ of size at most k is also a solution for G. Towards a contradiction suppose that X is not a solution for G. Thus, $G \setminus X$ has an obstruction K, containing the vertex c. Since $c \notin D$, by Lemma 9, we may assume that K is a chordless cycle of length at least 6. Next we will construct an obstruction in $(G \setminus \{c\}) \setminus X$, and thus contradict that X is a solution for $G \setminus \{c\}$.

Firstly, consider the case when K has no other vertex from C, apart from c. Let u and v be the neighbours of c in K. Lemma 19 implies that, if we manage to find a replacement for c from $C \setminus (X \cup V(K))$ (see Definition 17 and 18), then we can find an obstruction in $(G \setminus \{c\}) \setminus X$. Firstly consider the case when $u, v \in S$. Since $k+2$ common neighbours of u, v in C were marked by Marking Scheme 1, there is at least one common neighbour $c' \in (V(G) \setminus (X \cup \{c\})) \cap C$ which forms a replacement for c. This together with Lemma 19 and the premise of the case that K has no other vertex apart from c from C implies that $(G \setminus \{c\}) \setminus X$ contain an obstruction.

Recall that we are in the case when K contain exactly one vertex, namely, c from C. Next we suppose that $u \in S$ and $v \in V(A)$, for some $A \in \mathcal{A}$. Let s be the first vertex from S on the path from v to u in $K \setminus \{c\}$ (Clearly, such an s must exist since S is an approximate solution). Notice that every internal vertex on the path from c to s, containing v, on K must be from A. Consider the case when $u \neq s$. As K is a chordless cycle, we can obtain that $\{s, c\} \notin E(G)$, and thus $\{c, v_A\} \in E(H_s^1)$. Since c is unmarked, $c \notin V(M_s^1)$. Moreover, as M_s^1 is a maximum matching in H_s^1, it follows that $v_A \in V(M_s^1)$. Recall that Marking Scheme 2 marked $k+2$ vertices from $N(V(A)) \cap V_s^1$. Thus, there is a vertex $c' \in (C \setminus (X \cup \{c\})) \cap N(V(A)) \cap V_s^1$. By Lemma 15, since $\{v, c\} \in E(G)$ we must have $\{v, c'\} \in E(G)$. Marking Scheme 1 marked $k+2$ neighbors of $u \in S$ from C. Thus, there must exist a vertex $c_u \in C \setminus (X \cup \{c\})$, such that $\{c_u, u\} \in E(G)$. If $c' = c_u$, the vertex c' is a replacement for c with respect to K, and otherwise, the pair of vertices c', c_u forms a replacement for c in K. In either case, using Lemma 19 and the fact that K is a chordless cycle on at least 6 vertices, we can obtain an obstruction in $(G \setminus \{c\}) \setminus X$, for the case when $u \neq s$. Next we consider the case when $u = s$. Notice that $\{c, v_A\}$ is an edge in H_s^2. As c is unmarked, it must be the case that $v_A \in V(M_s^2)$. Since $k+2$ vertices in $N(V(A)) \cap C \cap N(s)$ were marked by Marking Scheme 2, there is a vertex $c' \in (N(V(A)) \cap N(s) \cap C) \setminus (X \cup \{c\})$. Again, by Lemma 15 we obtain that $\{v, c'\} \in E(G)$, and therefore the vertex c' forms a replacement for c, and hence from Lemma 19 we can construct an obstruction in $(G \setminus \{c\}) \setminus X$.

Next we consider the case when $u \in V(A_1)$ and $v \in V(A_2)$, for some (not necessarily distinct) $A_1, A_2 \in \mathcal{A}$, and K has no other vertex from C, apart from c. Let s_1, s_2 be the first vertices from S on the paths from u to v and v to u in $K \setminus \{c\}$, respectively. Observe that $\{c, v_{A_1}\} \in E(H_{s_1}^1)$ and $\{c, v_{A_2}\} \in E(H_{s_2}^1)$. Since c is unmarked, we must have $v_{A_1} \in M_{s_1}^1$ and $v_{A_2} \in M_{s_2}^1$. Marking Scheme 2 ensures that there are c_1', $c_2' \in C \cap V(G) \setminus (X \cup \{c\})$, where $c_1' \in N(V(A_1))$ and $c_2' \in N(V(A_2))$ which, by Lemma 15, are adjacent to vertices u, v respectively. If $c_1' = c_2'$, we use c_1' as a replacement for c in K, else the pair c_1', c_2' forms a replacement for c.

Now we consider the case when the chordless cycle K contains two vertices from C. (Note that since K is a chordless cycle, it can contain at most 2 vertices from C.) Since K contains c, exactly one of $u, v \notin C$, say $u \notin C$ (the other case can be argued symmetrically). We further consider the following cases based on whether $u \in S$. Firstly consider the case when $u \in S$. Since Marking Scheme 1 marked $k+2$ neighbours of $u \in S$ from C, it follows that there is a vertex $c' \in C \setminus (X \cup V(K))$ which is a replacement for c, and thus using Lemma 19 we can obtain an obstruction in $(G \setminus \{c\}) \setminus X$. Now suppose that $u \in V(A)$ for some $A \in \mathcal{A}$. Let s be the first vertex from S on the path from u to v in $K \setminus \{c\}$. This implies that $\{c, v_A\} \in E(H_s^1)$ and that $v_A \in V(M_s^1)$. Therefore, $k+2$ vertices in $N((V(A)) \cap V_s^1$ must have been marked by Marking Scheme 2. Since $|X| \leqslant k$, c is an unmarked vertex, and $|C \cap V(K)| \leqslant 2$, there must exist a marked vertex $c' \in (N(V(A)) \cap V_s^1) \setminus (X \cup V(K))$. By Lemma 15 we must have $\{u, c'\} \in E(G)$. Thus, c' forms a replacement for c in K, and we obtain an obstruction in $(G \setminus \{c\}) \setminus X$, using Lemma 19. ◀

We are now ready to prove Lemma 10.

Proof of Lemma 10. For the given instance (G, k) of Ptolemaic Deletion, an approximate solution S for G of size $\mathcal{O}(k)$, and a solution D for G that is 4-redundant with respect to a $(k+1)$-necessary family $\mathcal{W}$, returned by Lemma 6, we do the following. In polynomial time, we enumerate the set $\mathcal{C}$, of all maximal maximal cliques in $G \setminus S$ using Observation 14. For each clique $C \in \mathcal{C}$, we exhaustively apply Reduction Rule 1 and 2, where the lowest applicable reduction rule is applied first. Note that each of these reduction rules can be executed in polynomial time, and they can only be applied polynomially many times. If none of the reduction rules are applicable, then we argue below that the number of vertices in $C \setminus D$ can be bounded by $\mathcal{O}(k^3)$. Note that Marking Scheme 1 marks at most $\mathcal{O}(k^3)$ vertices in C. Note that while executing Marking Scheme 2, Reduction Rule 1 is not applicable. Thus the number of vertices marked by Marking Scheme 2 are bounded by $\mathcal{O}(k^3)$ as for every $s \in S$, there can be at most $k+1$ matching edges in both the matchings M_s^1 and M_s^2, and corresponding to each edge we mark $\mathcal{O}(k)$ vertices in C. As the unmarked vertices of $C \setminus D$ are deleted by Reduction Rule 2, we must have $|C \setminus D| \in \mathcal{O}(k^3)$. ◀

5 Obtaining Required Augmented Ptolemaic Deletion Instance and the Set R

The objective of this section is to prove Lemma 12. Let (G, k) be the given instance of Ptolemaic Deletion, where each maximal clique in G has $\mathcal{O}(k^3)$ vertices (except vertices in D). We further have, for G, an approximate solution S and a 4-redundant solution D, with respect to $\mathcal{W}$. We will create an instance (G', k', E_M, E_I) of Augmented Ptolemaic Deletion and a set $R \subseteq D$ (in polynomial time) that will satisfy the requirements of Lemma 12, or conclude that (G, k) is a no-instance of Ptolemaic Deletion. We initialize $G' := G$, $k' := k$, $E_M := \emptyset$, $E_I := \emptyset$, and $R := \emptyset$. We will maintain the following invariants throughout the execution of our algorithm.

Invariant 1: Item 1 to Item 5 of Lemma 12 are satisfied by (G', k', E_M, E_I) and R.

Invariant 2: Any $X \subseteq V(G')$ which intersects all chordless cycles in G' which contain no edge from E_I, intersects all chordless cycles in G.

By our initialization, the following observation easily follows.

▶ **Observation 24.** *After initialization, (G', k', E_M, E_I) and R satisfy Invariant 1 and 2.*

We will (recursively) modify (G', k', E_M, E_I) till it satisfies all the conditions of the lemma. We denote by $N_{\mathsf{rel}}(v)$ the relevant neighbours of a vertex v in $V(G')$, i.e., $N_{\mathsf{rel}}(v) = \{u : \{u, v\} \in E(G') \setminus E_I\}$. For a subset $Z \subseteq V(G')$, $N_{\mathsf{rel}}(Z) = (\cup_{v \in Z} N_{\mathsf{rel}}(v)) \setminus Z$. Each rule will take an Augmented Ptolemaic Deletion instance and output an equivalent instance of the problem that satisfies both Invariant 1 and 2. Since the initialization means that the input Ptolemaic Deletion is equivalent to the initial Augmented Ptolemaic Deletion instance, it follows that the finally obtained Augmented Ptolemaic Deletion instance is equivalent to the initial Ptolemaic Deletion instance.

For the sake of clarity, we reproduce a few results from [3] with modifications as necessary. We make use of the following rules which can be applied when necessary during the course of bounding the independent degree. We say that a rule is *safe* to apply, if the resulting instance of Augmented Ptolemaic Deletion and R satisfies Invariant 1 and 2, or it correctly declares that the instance of Augmented Ptolemaic Deletion is a no-instance.

▶ **Rule 1.** *Given two vertices $u \in S, v \in V(G)$ with the promise that every solution X to (G', k', E_M, E_I) contains either u or v, return $(G', k', E_M \cup \{\{u, v\}\}, E_I)$ and R.*

Clearly, the above rule is safe, as the resulting instance satisfies both Invariant 1 and 2.

▶ **Rule 2.** *If $k' < 0$, then declare the instance as a* NO *instance.*

Notice that if $k' < 0$, there can exist no solution (not even $\emptyset$), for the instance of AUGMENTED PTOLEMAIC DELETION. Thus, the above rule is safe.

▶ **Rule 3.** *If a vertex v is an endpoint of more than k edges in E_M, then return $(G' \setminus \{v\}, k' - 1, E_M \setminus \{\{v, u\} \mid u \in V(G')\}, E_I)$ and R.*

In the above rule, notice that such a v must necessarily belong to any solution for the AUGMENTED PTOLEMAIC DELETION instance (G', k', E_M, E_I). Moreover, notice that $(G' \setminus \{v\}, k' - 1, E_M \setminus \{\{v, u\} \mid u \in V(G')\}, E_I)$ and R satisfy both Invariant 1 and 2, and the above rule is safe.

▶ **Rule 4.** *If Rule 3 is not applicable and $|E_M| > k^2$ then declare the instance a* NO *instance.*

As Rule 3 is not applicable, each vertex in G' can be an endpoint of at most k pairs in E_M. Moreover, as $|E_M| > k^2$, and any solution of size at most k for (G', k', E_M, E_I) must intersect each set in E_M, the instance must be a NO-instance.

While bounding the independence degree following the algorithm of [3], we apply rules Rule 2 to Rule 4 whenever possible. However, we apply Rule 1 when we explicitly identify, during the course of the algorithm, two vertices u, v that satisfy the premise of the rule.

When we identify some pairs of vertices u, v such that at least one of them is included in any solution, we only add $\{u, v\}$ in E_M, but not in $E(G')$, in contrast to [3]. The addition of such an edge is not necessary in [3], but it can be safely added, since one of the endpoints of such an edge will anyway be deleted by the solution. We refrain from adding such edges because this may hamper the properties ensured by the 4-redundant solution D. This is why we choose to simply maintain the set E_M, rather than adding the edges to G'. Apart from the above minor change, we apply exactly the procedure as in Lemma 3.6 of [3] to reduce the independent degree of each vertex $v \in S$. Following the notation in [3], we note that the size of the approximate solution $f(k) = ck$ for PTOLEMAIC DELETION.

▶ **Lemma 25.** *In polynomial time, we can conclude either that (G', k', E_M, E_I) is a* NO-*instance of* AUGMENTED PTOLEMAIC DELETION*, or output an instance $\widehat{J} = (\widehat{G}, \widehat{k}, E_M, E_I)$ of* AUGMENTED PTOLEMAIC DELETION *where the independence degree $d_I^{G'}(v)$ of every $v \in S \cap V(G')$ is bounded by $\Delta = ck(k + 3)$. Furthermore, we can compute a set $\widehat{R} \subseteq D$, such that $\widehat{J}$ and $\widehat{R}$ satisfy Invariant 1 and 2.*

Proof. To reduce the independence degree of a vertex v [3] begins by computing a v-blocker, which is an approximate solution for G' that does not include v. In our case, we compute a v-blocker B_v as follows. Consider the WEIGHTED PTOLEMAIC DELETION instance with weight function w so that $w(v) = ck + 1$ and $w(u) = 1$ for every $u \in V(G') \setminus \{v\}$. If Approx returns a solution of size greater than ck, the vertex v must appear in any solution of size at most k, and hence we delete v and obtain a new instance $(G' \setminus \{v\}, k' - 1, E_M \setminus \{e \mid e \in E_M, e \cap v \neq \emptyset\}, E_I \setminus \{e \mid e \in E_I, e \cap v \neq \emptyset\})$. Else the solution returned by Approx is a v-blocker of size at most ck.

We remark that all the results and reduction rules in Section 3.3 of [3] hold for Ptolemaic graphs also. Since we use the same procedure for our result and all these results can be easily verified to hold for Ptolemaic graphs, we avoid re-stating and proving all the results for Ptolemaic graphs, instead we only present succintly the reasons why the results also hold for Ptolemaic graphs. All the structural results follow from the fact that Ptolemaic graphs are also chordal graphs. The reduction rules in [3] do one of the following: (i) identify a pair of vertices $\{u, v\}$, such that any solution must include one of u, v and add u, v to E_M, (ii) mark an existing edge as irrelevant, and add it to E_I. The procedure marks few edges $\{u, v\}$ as mandatory edges by identifying $k+2$ pairwise disjoint (chordless) cycles, except at $\{u, v\}$. Since chordless cycles are obstructions for Augmented Ptolemaic Deletion as well, the correctness of adding $\{u, v\}$ as a mandatory edge follows. Furthermore, whenever an edge is marked irrelevant, it is proved that any subset $X \subseteq V(G')$ hitting all chordless cycles containing only relevant edges also hits all chordless cycles containing the newly marked irrelevant edge. Hence we mimic exactly, the procedure in Section 3.3 of [3], to bound the independence degree of every vertex in S. From the result of Lemma 3.6 of [3], we bound the independence degree of every vertex $v \in S$ by $f(k)(k+3) = ck(k+3)$. Let $\widehat{J} = (\widehat{G}, \widehat{k}, E_M, E_I)$ be the resulting instance after applying the above procedure. Note that none of our rules till now changed the set R, which was initialized to $\emptyset$. We set $\widehat{R} = R = \emptyset$. The safeness of Rule 1 to 4 and the construction of $\widehat{R}$ implies that $\widehat{J}$ and $\widehat{R}$ satisfy Invariant 1 and 2. This concludes the proof. ◀

Hereafter we assume that the above lemma did not return that the instance is a no-instance of Augmented Ptolemaic Deletion. For notational simplicity, we let (G', k', E_M, E_I) and R be the instance and set returned by the lemma.

We will enhance R by adding vertices to it, so as to achieve Item 5 of Lemma 12. To this end, we introduce the following definition. A vertex v_c is a *private* vertex of a maximal clique C in a graph H if v_c is part of no other maximal clique in H. We now observe the following about private vertices in leaf bags of the inter-clique digraph of a given Ptolemaic graph.

▶ **Observation 26.** *For a Ptolemaic graph $\widehat{G}$ with inter-clique digraph $T_{\widehat{G}}$ the following hold:*

1. *The set of vertices $V(B)$ of $\widehat{G}$ in a bag B of $T_{\widehat{G}}$ is a maximal clique in G if and only if B has no incoming edges in $T_{\widehat{G}}$*
2. *A set of bags in $T_{\widehat{G}}$ can have a common vertex v if and only if they have a common descendant B_{desc} in $T_{\widehat{G}}$, such that B_{desc} contains v.*[6]

Proof. The first two items follow directly from the definition of inter-clique digraphs. Suppose the set of vertices of a bag B form a maximal clique in G and there is an edge directed from bag B' to bag B for some other bag B' of $T_{\widehat{G}}$. By construction, it must be the case that $V(B) \subseteq V(B')$. Since the vertices of B' also form a clique, the clique formed by vertices of B cannot be maximal. To prove the other direction, suppose a bag B has no incoming edges in $T_{\widehat{G}}$. Either $V(B)$ must be a maximal clique in $\widehat{G}$ or an intersection of certain maximal cliques of $\widehat{G}$. Now if $V(B)$ does not form a maximal clique, then there is a maximal clique in $\widehat{G}$ corresponding to some bag B' in $T_{\widehat{G}}$, such that $V(B) \subseteq V(B')$. But then there must be an edge directed from B' to B in $T_{\widehat{G}}$, a contradiction, proving the first part.

We now prove the second part. If bags $B_1, B_2 \ldots B_\ell$ have a common descendant B_{desc} containing v in $T_{\widehat{G}}$, we have $V(B_{\mathsf{desc}}) \subseteq V(B_i)$ for each $i \in [\ell]$, and the result follows. To prove the other direction, suppose $\bigcap_{i=1}^{\ell} V(B_i) = C$ for a clique C in $\widehat{G}$ with $v \in C$. There must exist a bag B corresponding to C in $T_{\widehat{G}}$. Since $V(B) \subseteq V(B_i)$ for each $i \in [\ell]$, there must be a directed path from each B_i to B. This proves the result. ◀

▶ **Observation 27.** *Consider a Ptolemaic Graph $\widehat{G}$, and a leaf bag B in its inter-clique digraph $T_{\widehat{G}}$. Then $V(B)$ forms a maximal clique in $\widehat{G}$, and must contain a private vertex v.*

[6] In a directed graph D and $u, v \in V(D)$, u is a descendant of v in D if there is a directed path from v to u in D.

Proof. Let B' be the only bag in $T_{\widehat{G}}$ adjacent to the leaf bag B in $Und(T_{\widehat{G}})$. We claim that the arc in $T_{\widehat{G}}$ must be directed from B to B' and not the other way. Suppose that this is not the case. Then $V(B) \subset V(B')$. It follows that $V(B)$ is not a maximal clique in $\widehat{G}$, and hence $V(B)$ must be the intersection of some number of maximal cliques $C_1, C_2 \dots C_l$ with $l \geqslant 2$. Therefore there must exist a directed path from the bags $B_1, B_2 \dots B_l$ corresponding to each of these maximal cliques to B in $T_{\widehat{G}}$. But then there also exists a directed path from each of $B_1, B_2 \dots B_l$ to B', which means that $V(B') \subseteq C_i$ for each $i \in [l]$. This contradicts the fact that $\bigcap_{i=1}^{l} C_i = V(B)$.

Since the arc in $T_{\widehat{G}}$ is directed from B to B', $V(B') \subseteq V(B)$. By Observation 26 $V(B)$ forms a maximal clique in $\widehat{G}$. Consider a vertex v in $V(B) \setminus V(B')$. v is not contained in $V(B')$. If v is contained in some other bag $B'' \neq B', B$, then $V(B'') \cap V(B) \neq \emptyset$, and therefore B'' and B must have a common descendant in $\widehat{G}$, say B_{desc} which contains v, by Observation 26. But the only outgoing edge from B is to B' and $v \notin V(B')$. Thus there cannot be a directed path from B to B_{desc} in $\widehat{G}$. ◀

We now apply the following procedure which we call BoundLeaves (repeatedly), for the instance (G', k', E_M, E_I) and the set R.

1. Compute the inter-clique digraph $T_{G' \setminus (R \cup S)}$ associated with the Ptolemaic graph $G' \setminus (R \cup S)$.
2. If there is a leaf bag B of $T_{G' \setminus (R \cup S)}$ with a private vertex $v \in V(B)$ in $G' \setminus (R \cup S)$, such that $v \notin N_{\mathsf{rel}}(S)$, do the following.
   a. If $v \in D$, add v to R.
   b. Else, set $G' := G' \setminus \{v\}$ and $E_I := E_I \setminus \{\{u, v\} \mid u \in V(G')\}$.
   c. Goto Step 1.
3. Return the instance (G', k', E_M, E_I) and the set R.

Again, we say that BoundLeaves is safe, if after every modification to R or G', the resulting instance of Augmented Ptolemaic Deletion and R satisfies Invariant 1 and 2.

▶ **Lemma 28.** *The procedure* BoundLeaves *is safe.*

Proof. First, we note that the vertices v picked by the procedure in Step 2 cannot be an endpoint of a mandatory edge. This is because every mandatory edge has an endpoint in S, and the vertices v considered during the algorithm do not have any relevant neighbour in S. Now suppose we apply Step 2 $\ell \geqslant 0$ times to obtain the reduced instance. Let $J_i = (G'_i, k', E_M, E_I^i)$ and R_i be the instance (G', k', E_M, E_I) and the set R after the execution of ith iteration, for each $i \in \{0, 1, 2 \dots \ell\}$. We will prove the lemma by induction on the iteration number. Note that at start of the procedure, we have the instance J_0 and the set R_0, which from Observation 24 and Lemma 25, satisfy Invariant 1 and 2. The above is the base case of our induction. Next consider iteration $i > 0$. For the induction hypothesis we assume that for each $0 \leqslant i' < i$, $J_{i'}$ and $R_{i'}$ satisfy Invariant 1 and 2. Let v be the vertex that was considered by Step 2 of the procedure, during iteration i. We consider the following cases based on whether $v \in D$.

1. Consider the case when $v \in D$. As $J_i = J_{i-1}$ for this case, items 1,2, and 4 of Lemma 12 are satisfied. Note that for the choice of v, in G_i, the set of relevant neighbours of v in the set $N_{\mathsf{rel}}(v) \setminus R_{i-1}$, forms a clique. Suppose that a chordless cycle K with no irrelevant edges contains v, then since $N_{\mathsf{rel}}(v) \setminus R_{i-1}$ is a clique, K must contain a vertex from R_{i-1}. But then, by the inductive hypothesis, K must contain an irrelevant edge - a contradiction. Thus we obtain that Item 3 of Lemma 12 is satisfied, and thus J_i and R_i

satisfy Invariant 1. Invariant 2 holds since $J_i = J_{i-1}$ and by the induction hypothesis, Invariant 2 holds upto the $(i-1)^{th}$ iteration.

2. Next consider the case when $v \notin D$. Note that the vertices in $N_{\mathsf{rel}}(v) \setminus R_i$ form a clique. We show that the instances $(G'_{i-1}, k', E_M, E_I^{i-1})$ and $(G'_i = G'_{i-1} \setminus \{v\}, k', E_M, E_I^i)$ are equivalent. It is sufficient to prove that any solution X of size at most k for $G'_{i-1} \setminus \{v\}$ is a solution for G'_{i-1}. Towards a contradiction, assume that this is not the case. Then $G'_{i-1} \setminus X$ contains an obstruction K which contains the vertex v. Since $v \notin D$, we may assume that K is a chordless cycle of length at least 6. Further, since any solution that intersects chordless cycles with relevant edges only also hits all chordless cycles, we may assume that K does not contain any irrelevant edge. Since $N_{\mathsf{rel}}(v) \setminus R_i$ forms a clique, K must contain a vertex from $R_i = R_{i-1}$. However, by the induction hypothesis, R_{i-1} must satisfy Invariant 1. In particular it satisfies Item 3 of Lemma 12, that is, any chordless cycle containing a vertex from R_{i-1} must contain an irrelevant edge. But then K must contain an irrelevant edge, a contradiction, since we started by assuming that K was a chordless cycle containing only relevant edges. It follows that the instances $(G'_{i-1}, k', E_M, E_I^{i-1})$ and (G'_i, k', E_M, E_I^i) are equivalent. Thus J_i and R_i satisfy Invariant 1. Invariant 2 holds since $E_I^i \subseteq E_i^{i-1}$.

◀

We are now ready to prove Lemma 12.

Proof of Lemma 12. We construct the inter-clique digraph of $G \setminus (R \cup S)$ in polynomial time, using Proposition 3. From Observation 24, Lemma 25 and 28, and non-applicability of Rule 3 and 4, to obtain a proof of the lemma, it is enough to argue that the number of leaves in the undirected inter-clique digraph of $G \setminus (R \cup S)$ is bounded as $\mathcal{O}(k^3)$. After applying procedure $\mathsf{BoundLeaves}$, every leaf bag B of $G \setminus (R \cup S)$ has the property that every private vertex $v_B \in B$ has a relevant neighbour in S. By Observation 27 there must exist at least one private vertex v_B in every leaf bag B. Taking one such vertex from each leaf bag B, we get an independent set. Since the independent degree of each vertex $v \in S$ is bounded by $\mathcal{O}(k^2)$, and the number of vertices in S is $\mathcal{O}(k)$, we obtain the desired bound. ◀

6 Obtaining Desired Augmented Ptolemaic Deletion Instance

The objective of this section is to prove Lemma 13. Let (G', k', E_M, E_I) be the instance of Augmented Ptolemaic Deletion and R be the set returned by Lemma 12. We denote by H the Ptolemaic graph $G' \setminus (R \cup S)$. Let T_H denote the inter-clique digraph of H. The number of leaves of T_H is $\mathcal{O}(k^3)$, and hence the number of degree $\geqslant 3$ bags of $Und(T_H)$ is $\mathcal{O}(k^3)$ as well. We now turn to bounding the number of degree 2 bags in $Und(T_H)$. We assume that in G', each maximal clique has at most $\mathcal{O}(k^3)$ vertices, excepting those in D (see Lemma 10 and 12).

First, we observe the following basic property of inter-clique digraphs:

▶ **Observation 29.** *Given a Ptolemaic graph $\widehat{G}$, any 2-degree bag of $Und(T_{\widehat{G}})$ must either have both its edges in $T_{\widehat{G}}$ as outgoing edges, or have both edges as incoming edges. Consequently, as we traverse an undirected path in $T_{\widehat{G}}$ where each bag is of degree 2, edges alternate in direction, i.e. consecutive bags B_1, B_2 along the path have the property that if B_1 has both of its edges in $T_{\widehat{G}}$ as incoming (resp. outgoing) edges, then B_2 has both of its edges in $T_{\widehat{G}}$ as outgoing (resp. incoming) edges.*

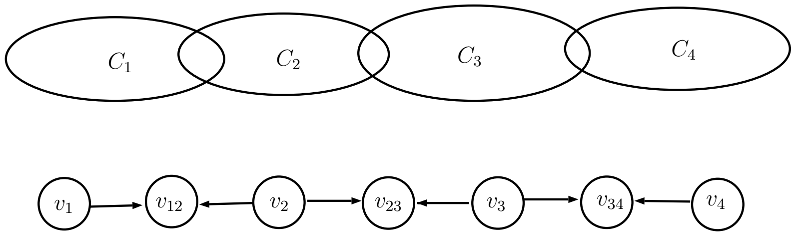

Figure 2 A 2-degree path in the inter-clique digraph of a Ptolemaic Graph and the corresponding 2-degree clique sequence. Vertex v_i in the inter-clique digraph corresponds to maximal clique C_i. The vertices v_{ij} correspond to intersections of cliques C_i and C_j. The bags v_1 and v_4 also have degree 2 in the undirected inter-clique digraph, their other neighbours are outside the 2-degree path.

Proof. Suppose that a degree 2 bag B has one incoming and one outgoing edge. Let B' be the bag which has an outgoing edge to B. By Observation 26 $V(B)$ cannot be a maximal clique in $\widehat{G}$ and therefore B must correspond to the intersection of a certain number of (at least 2) maximal cliques of $\widehat{G}$. That is, there is a set of maximal cliques $\widehat{\mathcal{C}}$ of $\widehat{G}$ with size at least 2 so that $V(B) = \bigcap_{C \in \widehat{\mathcal{C}}} C$. By the construction of the inter-clique digraph $T_{\widehat{G}}$, the bag B must be a common descendant of the bags corresponding to the maximal cliques in $\widehat{\mathcal{C}}$. However, the bags corresponding to these maximal cliques in $T_{\widehat{G}}$ must then also have B' as a common descendant. Since $V(B) \subset V(B')$, this means that these two maximal cliques intersect in more vertices than those in bag B, a contradiction since we started with the assumption that $V(B)$ is the exact set of vertices in which these maximal cliques intersect. The second part easily follows from its first part. ◀

Consider a maximal path in $Und(T_H)$, where each bag is of degree 2 in $Und(T_H)$. Let us call the two bags at the ends of this path (that is, bags of T_H with degree 1 in the path) as *terminal* bags. By Observation 29, it follows that each of the terminal bags must have either both edges incoming, or both edges outgoing in T_H. Suppose either terminal bag has both its edges as incoming edges. Then we remove this bag from this path and obtain a new (shorter) path where both the terminal bags have both edges outgoing in T_H. Notice that the vertices of G' present in the removed terminal bags are already accounted for in the newly obtained path, since the removed bag has an in-neighbour in the path. At this point, we have a path P in T_H which alternates between incoming and outgoing edges, each bag on P has degree 2 in $Und(T_H)$ and the bags at the end of this path have both edges outgoing in T_H. We call such a path a 2-*degree path*. Since the number of leaves of T_H is $\mathcal{O}(k^3)$, the number of 2-degree paths P is bounded by $\mathcal{O}(k^3)$.

A *maximal* 2-degree path is a 2-degree path which cannot be extended further on either side (without violating the property of a 2-degree path). During the course of our algorithm, we mark certain bags in T_H. We call a 2-degree path *unmarked* if none of the bags on the path are marked. Further, by a *maximal unmarked* 2-degree path, we mean a 2-degree path which (i) does not have any marked bags, and (ii) cannot be extended without violating the property of a 2-degree path, while ensuring every bag is unmarked. Note that these definitions change with the progress of the algorithm, since the set of marked bags changes. Henceforth whenever we refer to "marked" bags, we assume that all the marking schemes for bags prior to the reference have been applied.

We now have the following observation about vertices in 2-degree paths, whose proof directly follows from item 2 of Observation 26 and Observation 29.

▶ **Observation 30.** *Given a Ptolemaic graph $\widehat{G}$ with inter-clique digraph $T_{\widehat{G}}$,*

1. *A vertex $v \in V(\widehat{G})$ can be contained in at most 3 bags of a given 2-degree path. Furthermore, the set of bags in which v appears must be consecutive in the path.*
2. *If $v \in V(\widehat{G})$ is present in more than one maximal 2-degree path, then it can appear only in the terminal bags of both the 2-degree paths.*

Proof. By Observation 29, a 2-degree path alternates between incoming and outgoing edges. If a vertex v is contained in two different bags B_1 and B_2 of a 2-degree path, Observation 26 gives us that B_1 and B_2 must have a common descendant in $T_{\widehat{G}}$. This can happen in one of two cases.

- ▷ There is a bag B_3 on the 2-degree path such that both B_1 and B_2 have an outgoing edge to B_3. Since B_3 has no outgoing edges and does not have any descendants, it follows from Observation 26 that v can only be contained in the bags B_1, B_2 and B_3 on the 2-degree path.
- ▷ B_1 has an outgoing edge to B_2. Call the other in-neighbour of B_2 as B_3. Again, since B_2 has no outgoing edges, v can only be contained in bags B_1, B_2 and B_3 on the 2-degree path. More precisely, if the bag B_3 lies on P, then v is contained in 3 bags, else it is contained in 2 bags of P.

In either case, it is clear that the set of bags of P in which v appears are consecutive on P.

To prove the second part, note that by the definition of a 2-degree path, both the terminal bags of a 2-degree path have both their edges as outgoing edges in $T_{\widehat{G}}$. By Observation 26, if a vertex v is present in two bags from two different maximal 2-degree paths P_1 and P_2, the two bags must have a common descendant in $T_{\widehat{G}}$. This means that v cannot be present in a non-terminal bag of either path, since non-terminal bags on a 2-degree path P do not have descendants outside the path P. ◀

Using the properties in Observation 26 and Observation 29 we observe the following.

▶ **Observation 31.** *A maximal 2-degree path P in T_H corresponds to a sequence $\mathcal{C}(P) = (C_1, C_2 \ldots C_l)$ of maximal cliques in H, where $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$ for each $i, j \in [l]$.*

Proof. Notice that every bag on the path P which has two outgoing edges (and thus, no incoming edges) must correspond to a maximal clique in G by item 1 of Observation 26. Thus, as we traverse the path from one end to the other, every alternate bag on P corresponds to a maximal clique, and a bag B with two incoming edges corresponds to the intersection of the maximal cliques corresponding to the bags B_1 and B_2 to the immediate left and right of B. By item 2 of Observation 26, two maximal cliques intersect if and only if they have a common descendant in T_H. It follows that a bag B' on the path P corresponding to a maximal clique intersects only with atmost two other maximal cliques, (corresponding to one bag on the left and one on the right in P, both at distance 2 from B'). See Figure 2 for a clear picture.

Listing the maximal cliques based on the order in which their corresponding bags appear in P from one end of P to the other, we get an ordering $(C_1, C_2 \ldots C_l)$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$ for $i, j \in [l]$.

◀

Let us call the sequence of maximal cliques $\mathcal{C}(P)$ in Observation 31 as the *2-degree clique sequence* associated with the path P.

▶ **Lemma 32.** *Consider a maximal* 2*-degree path* P *in* T_H *and the corresponding 2-degree clique sequence* $\mathcal{C}(P) = (C_1, C_2 \dots C_l)$*. Consider a vertex* $v \in C_i$ *for some* $i \in \{2, 3 \dots l-1\}$*. Then* v *is part of at most* 2 *maximal cliques in* H*, and not part of any maximal clique outside* $\mathcal{C}(P)$ *in* H*.*

Proof. Clearly, v is part of atmost two maximal cliques in $\mathcal{C}(P)$ as two cliques C_i,C_j intersect iff $|i-j| \leqslant 1$. Suppose v is part of some other maximal clique C' in H. Consider the bags B_i, B' corresponding to C_i, C' in T_H. Then B' and B_i must have a common descendant in T_H. But this is impossible since the bag B' lies outside the path P, and the terminal bags of P have both their edges as outgoing edges in T_H.

◀

▶ **Lemma 33.** *Consider a maximal* 2*-degree path* P *in* T_H *with* $\mathcal{C}(P) = (C_1, C_2 \dots C_l)$*. Consider vertices* $u \in C_i$ *and* $v \in C_j$ *for some* $i, j \in \{2, 3 \dots l-1\}$*. If* $(u, v) \in E(G')$ *then* u *and* v *must both appear in bag* B *for some bag* B *on the path* P*.*

Proof. Consider such an edge $(u, v) \in E(G')$ such that $u \in C_i$ and $v \in C_j$. Notice that the edge (u, v) must appear in some maximal clique of H, say C'. Suppose $C' \in \mathcal{C}(P)$. Then clearly, u and v both appear in the bag B' corresponding to C'. B' must lie on P since $C' \in \mathcal{C}(P)$.

Therefore we may assume $C' \notin \mathcal{C}(P)$. But then v is part of a maximal clique outside $\mathcal{C}$, which contradicts the conclusion of Lemma 32.

◀

During the course of bounding the number of vertices contained in bags of 2-degree paths of T_H, we mark a “small” number of bags in T_H, and bound the number of vertices in maximal unmarked 2-degree paths. We start by marking 2 bags from both the ends of every maximal 2-degree path.

▶ **Marking Scheme 3.** *For every* 2*-degree path, mark the* 2 *leftmost and* 2 *rightmost bags (If there are fewer than* 4 *bags on the path, mark all the bags).*

▶ **Observation 34.** *After applying Marking Scheme 3, a vertex* v *can be present in bags of atmost one maximal unmarked* 2*-degree path* P*. Further,* v *may appear in atmost* 3 *bags of this* 2*-degree path, and the set of bags on* P *in which* v *appears must be consecutive on* P*.*

Proof. Follows directly from Observation 30. ◀

▶ **Observation 35.** *After applying Marking Scheme 3, consider a* 2*-degree path* P *and the corresponding* 2*-degree clique sequence* $\mathcal{C}(P) = (C_1, C_2 \dots C_l)$*. Then the result of Lemma 33 holds for vertices* u, v *even when* $u, v \in C_1 \cup C_l$*.*

Proof. Follows directly once we note that marking 2 bags from the left and right of a path P removes the first and last maximal cliques in the corresponding 2-degree clique sequence.

◀

▶ **Observation 36.** *Consider a vertex* $v \in V(B)$ *for some bag* B *on a maximal* 2*-degree path* P*. Then every vertex* $u \in C_j$ *for some* $C_j \in \mathcal{C}(P)$ *such that* $(v, u) \in E(G')$ *must be contained in one among* 11 *bags on* P *- more precisely, in* B *or the* 5 *bags to the left and right of* B *in* P*.*

Proof. By Observation 34 both v and u can be contained in atmost 3 bags on P, and these must be consecutive on P. Further, by Observation 35 and Lemma 33, there must be a bag B' on P such that both u and v appear in $V(B')$, and the result follows. ◀

Since there are $\mathcal{O}(k^3)$ many 2-degree paths, Marking Scheme 3 marks $\mathcal{O}(k^3)$ number of bags.

We now use the following marking scheme.

▶ **Marking Scheme 4.** *Consider a vertex v which is adjacent to a vertex in S via a relevant edge, i.e. $(s, v) \in E(G')$ for some $s \in S$. For every bag B across all maximal 2-degree paths P, mark B if v is contained in B.*

▶ **Lemma 37.** *Exhaustive application of Marking Scheme 4 marks $\mathcal{O}(k^3)$ bags with a total of atmost $\mathcal{O}(k^6)$ many vertices.*

Proof. By Observation 34 every invocation of Marking Scheme 4 marks atmost 3 bags and atmost $\mathcal{O}(k^3)$ many vertices (excluding vertices from D). Using Observation 36 there are atmost 10 other bags across all 2-degree paths in which neighbours of v_B may be contained. Recall that the independence degree $|N_{\mathsf{rel}}(s)|$ of every vertex $s \in S$ is bounded as $\mathcal{O}(k^2)$, and there are only $\mathcal{O}(k)$ vertices in S. Thus Marking Scheme 4 can be invoked at most $\mathcal{O}(k^3)$ times, and thus mark a total of atmost $\mathcal{O}(k^6)$ vertices across all invocations. ◀

On a high level, we next show that the $\mathcal{O}(k^3)$ size bound for clique size obtained earlier can be improved to $\mathcal{O}(k)$ for each clique in $\{C_2, C_3 \dots C_{l-1}\}$. We do this by marking a few vertices, and deleting unmarked vertices.

▶ **Marking Scheme 5.** *For each $i \in \{2, 3, \dots l\}$, mark $\min\{|C_i \cap C_{i-1}|, k+3\}$ vertices in $C_i \cap C_{i-1}$.*

▶ **Reduction Rule 3.** *Given an instance (G', k', E_M, E_I) of* Augmented Ptolemaic Deletion, *and a 2-degree clique sequence $\mathcal{C}(P) = \{C_1, C_2, \dots C_l\}$ corresponding to a 2-degree path P, for any $i \in \{2, 3, \dots l-1\}$, delete any vertex $v \in C_i \setminus D$ unmarked by Marking Scheme 5 without reducing the parameter to obtain the instance $(G' \setminus \{v\}, k', E_M, E'_I)$, where $E'_I = E_I \setminus \{\{a, v\} \mid a \in V(G')\}$.*

▶ **Lemma 38.** *Reduction rule 3 is safe.*

Proof. We first prove the forward direction. Consider a solution X to (G', k', E_M, E_I). Clearly, $G' \setminus X$ is obstruction-free. The set E_M has remained unchanged (this is because only relevant edges with an endpoint in s can be mandatory), and $E'_I \subset E_I$. X is therefore also a solution to $(G' \setminus \{v\}, k', E_M, E'_I)$.

We now prove the reverse direction. Let X be any solution to $(G' \setminus \{v\}, k, E_M, E'_I)$. Then we claim that X is also a solution to (G', k, E_M, E_I). It suffices to prove that $G' \setminus X$ does not contain an obstruction. Assume to the contrary, that there is an obstruction. Since $v \notin D$, this obstruction must be a chordless cycle K of length at least 6 that contains v. Further we may assume that K contains only relevant edges, since we have the invariant that any subset $X \subseteq V(G')$ that hits all chordless cycles with no irrelevant edge also hits all chordless cycles with irrelevant edges. By Lemma 12 K cannot contain any vertex from R, since any chordless cycle that contains a vertex from R must contain an irrelevant edge. Further, since we are looking at a vertex in a bag which is unmarked by Marking Scheme 4, both the neighbours a and b of v in K must be from $V(H)$ (Recall that $H = G' \setminus (R \cup S)$). Notice that by Lemma 32 v maybe part of atmost two maximal cliques of H. If it is part of exactly one maximal clique (only C_i), then its neighbourhood in H forms a clique, and such a K cannot exist.

Henceforth we may assume v is part of two maximal cliques C_j and C_{j+1}, $j \in \{i-1, i\}$, in H. We may also assume without loss of generality, $a \in C_j$, $b \in C_{j+1}$. Now consider the cycle K' obtained by replacing in K, the path avb by the path $av'b$, where v' is a marked vertex in $(C_j \cap C_{j+1}) \setminus (X \cup \{a, b\})$. Since v was an unmarked vertex, and $k + 3$ vertices should have been marked in $C_j \cap C_{j+1}$, such a v' must exist since $|X| \leqslant k$. Observe that the cycle K' must be chordless, since the vertices v' and v are twins in H (they are both contained in exactly two maximal cliques C_j and C_{j+1} of H). But then K' forms an obstruction in $G' \setminus (X \cup \{v\})$, a contradiction.

◀

We apply Reduction rule 3 exhaustively and again apply Marking Scheme 3 to mark 2 bags from either end of a maximal unmarked 2-degree path. Note that Marking Scheme 3 marks $\mathcal{O}(k^3)$ bags, since the number of maximal unmarked 2-degree paths is $\mathcal{O}(k^3)$. We have the following observation whose correctness is immediate.

▶ **Observation 39.** *After exhaustive application of Reduction rule 3 followed by Marking Scheme 3, for any 2-degree path P, the size of any maximal clique in $\mathcal{C}(P)$ is bounded as $\mathcal{O}(k)$, except for vertices in D.*

Proof. Consider a path P before applying Marking Scheme 3 with $\mathcal{C}(P) = (C_1, C_2 \dots C_l)$. Since Reduction rule 3 has been applied exhaustively, the size of each clique C_i for $i \in \{2, 3 \dots l-1\}$ is bounded by $\mathcal{O}(k)$ (except for vertices in D). After marking 2 bags from either end of P, call the new (trucated) maximal unmarked 2-degree path P'. Clearly, $\mathcal{C}(P') = (C_2, C_3 \dots C_{l-1})$, and hence the size of any maximal clique in $\mathcal{C}(P')$ is bounded by $\mathcal{O}(k)$ (except for vertices in D).

◀

Fix a maximal unmarked 2-degree path P, and suppose $\mathcal{C}(P) = (C_1, C_2 \dots C_l)$. Roughly speaking, the goal of the next reduction rule is to mark a small number of bags so that each resulting unmarked maximal subpath does not have vertices adjacent to vertices of the approximate solution.

▶ **Reduction Rule 4.** *Given an instance (G', k', E_M, E_I), for a vertex $s \in S$ and a maximal unmarked 2-degree path P, if s is adjacent to vertices in more than $4k+4$ cliques of $\mathcal{C}(P)$, then delete s and decrease the parameter by 1, to obtain the instance $(G' \setminus \{s\}, k'-1, E_M \setminus E^s_M, E_I \setminus E^s_I)$, where $E^s_M = \{\{s,v\} \,|\, v \in V(G'), \{s,v\} \in E_M\}$ and $E^s_I = \{\{s,v\} \,|\, v \in V(G'), \{s,v\} \in E_I\}$.*

▶ **Lemma 40.** *Reduction rule 4 is safe.*

Proof. Consider vertices v_i and v_j from two distinct cliques $C_i, C_j \in \mathcal{C}(P)$ with $C_i \cap C_j = \emptyset$ such that $(v_i, s) \in E(G')$ and $(v_j, s) \in E(G')$. Consider a shortest path P' between v_i and v_j, all of whose internal vertices are from $\cup_{t=i}^{j} C_t$. The vertex s together with v_i, v_j and the path P' form a cycle K of length at least 5. Since P' is a shortest path, any chord in K must involve the vertex s, and therefore by Observation 16, $G'[V(K)]$ contains an obstruction.

If s is adjacent to vertices from at least $4k+4$ cliques from $\mathcal{C}$, since $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ we obtain an independent set $I = \{v_{i_1}, v_{i_2} \dots v_{i_{2k+2}}\}$ in $\cup_{t=1}^{l} C_t \cap N(s)$ where

$v_{i_t} \in C_{i_t}$ for every $t \in [2k+2]$, and $i_1 < i_2 \dots i_{2k+1} < i_{2k+2}$. Applying the above argument to the $k+1$ pairs of vertices $(v_{i_{2t-1}}, v_{i_{2t}})$ for every $t \in [k+1]$ we obtain $k+1$ obstructions which intersect only at s. It follows that s must be part of any solution of size atmost k. ◀

▶ **Marking Scheme 6.** *Consider a maximal unmarked 2-degree path. For every bag B on this path, if there is a vertex $v_B \in V(B)$, so that v_B is adjacent to some vertex $s \in S$, mark B.*

After exhaustively applying Reduction rule 4, notice that since there are $\mathcal{O}(k)$ vertices in S, for every maximal unmarked 2-degree path P there are $\mathcal{O}(k^2)$ many cliques in $C(\mathcal{P})$ which may have a vertex adjacent to some vertex in s. Consequently, there are $\mathcal{O}(k^2)$ bags on P which contain a vertex adjacent to some vertex in S. Since there are $\mathcal{O}(k^3)$ many maximal unmarked 2-degree paths, Marking Scheme 6 marks atmost $\mathcal{O}(k^5)$ bags, each of which contain $\mathcal{O}(k)$ vertices of $G' \setminus D$, and thus a total of $\mathcal{O}(k^6)$ vertices.

Next, to account for the redundant solution we employ the following scheme.

▶ **Marking Scheme 7.** *Mark each bag B of a maximal unmarked 2-degree path, if there is a vertex v_B in B so that (a) $\exists d \in D$ so that $(v_B, d) \in E(G')$ or (b) $v_B \in D$.*

▶ **Observation 41.** *Marking Scheme 7 marks $\mathcal{O}(k^5)$ bags which may have a total of at most $\mathcal{O}(k^6)$ vertices.*

Proof. By Item 5 of Lemma 12 the neighbourhood of each vertex of R in H is a clique. Thus a vertex r of R can be adjacent to vertices present in bags of only one among all maximal unmarked 2-degree paths. Further, by Observation 36 it follows that the neighbours of r contained in bags of the path P can be contained in at most 11 bags of P, for a total of $\mathcal{O}(k)$ vertices (excepting vertices in D). Also, for any bag B on a maximal unmarked 2-degree path, no vertex $v_B \in B$ can have a neighbour in S after exhaustively applying Marking Scheme 6. Every vertex in $D \setminus (R \cup S)$ is contained in $G' \setminus (R \cup S)$. Therefore the total number of bags and vertices marked by Marking Scheme 7 are $\mathcal{O}(k^5)$ and $\mathcal{O}(k^6)$, since $|D| \in \mathcal{O}(k^5)$. ◀

After exhaustive application of Reduction rule 4 and Marking Scheme 7 to account for redundant solution vertices, notice that every vertex present in bags of a maximal unmarked 2-degree path in $G' \setminus (R \cup S)$ is not adjacent to any vertex in $R \cup S$. We now reduce the number of vertices in such paths.

To avoid gems interfering in the subsequent analysis, we again apply Marking Scheme 3.

▶ **Observation 42.** *Consider a maximal unmarked 2-degree path P before applying Marking Scheme 3, and denote the set of vertices of G' present in bags of this path as $V(P)$. Suppose a vertex $v \in V(P)$ is part of a gem in G'. Then v cannot be contained in an unmarked 2-degree bag after applying Marking Scheme 3.*

Proof. The vertex v participates in a gem obstruction, say $\mathbb{O}$. But $\mathbb{O}$ must contain a vertex $s \in S$. It follows that there is a path of length atmost 2 from s to v in G'. Note that $(s, v) \notin E(G')$, else it would have been marked by Marking Scheme 6. Therefore, there must be a vertex w such that $(v, w) \in E(G')$, $(w, s) \in E(G')$. Also, $w \notin V(P)$ again by the previous argument. Note that $w \in V(H)$ since no vertex in a maximal unmarked 2-degree path is adjacent to any vertex in $R \cup S$ after application of Reduction rule 4 and Marking Scheme 7.

Let $\mathcal{C}(P) = (C_1, C_2, \dots C_l)$. Consider a maximal clique C_{vw} in H that includes both the vertices v, w. Since $w \notin V(P)$, $C_{vw} \notin \mathcal{C}$. Using Lemma 32, this implies that $v \notin C_i$ for each

$i \in \{2, 3, \ldots l-1\}$. After applying Marking Scheme 3, let us denote the truncated maximal 2-degree path obtained from P as P'. Then $\mathcal{C}(P') = (C_2, C_3 \ldots C_{l-1})$. But $v \notin C_i$ for each $i \in \{2, 3, \ldots l-1\}$, and thus cannot be contained in any bag of P'. ◀

In this process we mark $\mathcal{O}(k^5)$ bags, and thus $\mathcal{O}(k^6)$ vertices since the size of each maximal clique in any unmarked 2-degree bag is bounded by $\mathcal{O}(k)$. Notice that the the number of unmarked maximal 2-degree paths is still $\mathcal{O}(k^5)$. At this point no vertex in an unmarked maximal 2-degree path can be part of a gem in G.

Now consider the 2-degree clique sequence $\mathcal{C} = (C_1, C_2 \ldots C_l)$ that corresponds to a maximal unmarked 2-degree path P, such that $l \geqslant 5$. Let us denote the subgraph of G' induced on the vertices $\cup_{i=1}^{l} C_i$ as G_P^*. Consider the augmented graph $\widehat{G}_P = (V(\widehat{G}_P) = V(G_P^*) \cup \{s, t\}, E(\widehat{G}_P) = E(G_P^*) \cup \{(s, v) \ : \ v \in C_1\} \cup \{(v, t) \ : \ v \in C_l\})$. Before proceeding to our next reduction rule, we note that since no vertex in $V(G_P^*)$ is adjacent to vertices in $R \cup S$, and every mandatory edge has an endpoint in S, no vertex in G_P^* can be an endpoint of some mandatory edge in E_M. Further, Marking Scheme 7 ensures that for every vertex $v \in V(G_P^*)$, $v \notin D$.

Roughly speaking, our next reduction rule is based on the idea that it is sufficient to "preserve" the size of a minimum s-t separator in $\widehat{G}_P$.

▶ **Reduction Rule 5.** *For an unmarked maximal 2-degree path P with $|C(P)| = \{C_1, C_2 \ldots C_l\}$ and $l \geqslant 5$, construct the graphs G_P^* and $\widehat{G}_P$. Let s_{min} be the size of a minimum s-t separator in $\widehat{G}_P$.*[7] *Construct a graph G'' by replacing in G' the vertices $C_2 \cup C_3 \ldots C_{l-1}$ by a single clique C_{min} of size s_{min}. Choose s_{min} vertices each from $C_1 \cap C_2$ and $C_l \cap C_{l-1}$* [8]*, and add new edges from every chosen vertex to every vertex of C_{min}. Obtain the reduced instance as (G'', k', E_M, E_I).*

Let us denote the modified graph obtained from G' as G'', and the original set of vertices in the 2-degree clique sequence as $M' = \cup_{i=1}^{l} C_i$. We have the following observation.

▶ **Observation 43.** *Consider the augmented graph $\widehat{G''}$ obtained from G'' by adding vertices s and t, adding edges from s to all vertices in C_1, and adding edges from t to all vertices in C_l. Then the size of a minimum $s-t$ separator in $\widehat{G''}$ is s_{min}.*

Proof. Clearly, C_{min} is a $s-t$ separator of size s_{min} in $\widehat{G''}$. Now suppose that the minimum $s-t$ separator X in $\widehat{G''}$ has size less than s_{min}. Notice that $|C_1 \cap C_2|, |C_{l-1} \cap C_l| \geqslant s_{min}$. Also $|C_{min}| = s_{min}$. This means that $\widehat{G''} \setminus X$ contains at least one vertex from each of $C_1 \cap C_2$, $C_{l-1} \cap C_l$ and C_{min}. But then there is a path from s to t in $\widehat{G''} \setminus X$, a contradiction. ◀

▶ **Lemma 44.** *Reduction rule 5 is safe.*

Proof. After the modification, we denote the set of vertices $C_1 \cup C_l \cup C_{min}$ as M''. We first prove the forward direction. Let X' be any solution of size at most k' for G'. Now consider the solution X'' which is obtained from X' as follows. If $|X' \cap M'| < s_{min}$, then $X'' = X' \setminus M'$. Otherwise, $X'' = (X' \setminus M') \cup C_{min}$. Clearly, $|X''| \leqslant k'$. If X'' is not a solution for G'', then there is an obstruction K'' in $G'' \setminus X''$. Notice that this obstruction cannot be a gem, due to Marking Scheme 42. Thus, K'' is a chordless cycle, and further K'' must contain at least one vertex from M''.

[7] An s-t separator of minimum size can be found in polynomial time (Chapter 7, [18]).
[8] Notice that $|C_1 \cap C_2|, |C_l \cap C_{l-1}| \geqslant s_{min}$

Any chordless cycle K'' which uses a vertex from M'', must enter and exit M'' exactly once from the rest of the graph, and that too it must enter through a vertex from C_1 and exit through C_l (or vice versa). In particular, K'' intersects M'' in an induced path P'' of G''. Suppose that P'' intersects C_1 and C_l in v_1 and v_l respectively. The existence of P'' implies that X'' does not include C_{min}, and hence the solution X' must satisfy $|X' \cap M'| < s_{min}$. This in turn means that there is a path P' in $M' \setminus X'$, which goes through v_1 and v_l that survives in $G' \setminus X'$. Now consider the cycle K' that is obtained by replacing in K'', the path P'' (in G'') by the path P' (in G'). Since $l \geqslant 5$, it follows that there are at least 5 vertices on P'. K' is then an obstruction in $G' \setminus X'$, a contradiction.

Now we prove the backward direction. Let X'' be any solution of size at most k' for G''. Now consider the solution X' which is obtained from X'' as follows. If $|M'' \cap X''| \geqslant s_{min}$, then $X' = (X'' \setminus M'') \cup T$, where T is any minimum $s - t$ separator in $\widehat{G_P}$. Otherwise, $X' = X'' \setminus M''$. Clearly, $|X'| \leqslant k'$. If X' is not a solution for G', then there is an obstruction K' in $G' \setminus X'$. Notice that this obstruction cannot be a gem. Thus, K' is a chordless cycle, and further K' must contain at least one vertex from M'. Any chordless cycle K' which uses a vertex from M', must enter and exit M' exactly once from the rest of the graph, and that too it must enter through a vertex from C_1 and exit through C_l (or vice versa). In particular, K' intersects M' in an induced path P' of G'. Suppose that P' intersects C_1 and C_l in v_1 and v_l respectively. The existence of P' implies that X' does not include a minimum $s - t$ separator, and hence the solution X'' satisfies $|X'' \cap M''| < s_{min}$. This in turn means that there is a path P'' in $M'' \setminus X''$, which goes through v_1 and v_l that survives in $G'' \setminus X''$. Now consider the cycle K'' that is obtained by replacing in K', the path P' by the path P''. By construction, this path is of length at least 2. It follows that K'' is then an obstruction in $G'' \setminus X''$, a contradiction. ◀

Let $(\widetilde{G}, \widetilde{k}, \widetilde{E}_M, \widetilde{E}_I)$ denote the instance of AUGMENTED PTOLEMAIC DELETION after application of all the above reduction rules. We now prove Lemma 13.

Proof of Lemma 13. Since all the reduction rules are correct, the instances are equivalent. Since every reduction rule either keeps the parameter the same or decreases it, $\widetilde{k} \leqslant k$. We do not add any mandatory edges, therefore $|\widetilde{E}_M| \leqslant |E'_M|$. Now we show that $|V(\widetilde{G})| \in \mathcal{O}(k^6)$.

Firstly, the size of any maximal clique in $G' \setminus (S \cup D)$ was bounded by $\mathcal{O}(k^3)$, and not accounting for the redundant solution, the size of each bag in the inter-clique digraph $T_{G'}$ corresponding to the graph $G' \setminus (R \cup S)$ was bounded by $\mathcal{O}(k^3)$. After bounding the number of leaves in the $T_{G'}$, the total number of vertices in degree-1 and degree $\geqslant 3$ bags of $T_{G'}$ was bounded by $\mathcal{O}(k^6)$.

At this stage, there were $\mathcal{O}(k^3)$ 2-degree paths in $T_{G'}$. Across all such paths we marked $\mathcal{O}(k^3)$ many bags (bags in which vertices had relevant neighbours in S), for a total of $\mathcal{O}(k^6)$ vertices. We now reduced the size of every clique in unmarked 2-degree paths to $\mathcal{O}(k)$ (excluding vertices from D). Next, we marked $\mathcal{O}(k^2)$ bags on each path, for a total of $\mathcal{O}(k^5)$ bags, and thus $\mathcal{O}(k^6)$ vertices to account for edges from S to vertices in 2-degree paths (excluding vertices from D). We further marked $\mathcal{O}(k^5)$ bags in 2-degree paths to account for the redundant solution, for a total of $\mathcal{O}(k^6)$ vertices. Finally, we showed that a maximal unmarked 2-degree path whose corresponding 2-degree clique sequence has at least 5 maximal cliques can now be replaced by $\mathcal{O}(k)$ many vertices. Since the number of maximal unmarked 2-degree paths is $\mathcal{O}(k^5)$, again, $\mathcal{O}(k^6)$ vertices are present in such paths. The size of the redundant solution and the approximate solution together account for $\mathcal{O}(k^5)$ vertices, giving us $|V(\widetilde{G})| \in \mathcal{O}(k^6)$ vertices. ◀

References

1 Akanksha Agrawal, Sushmita Gupta, Pallavi Jain, and R. Krithika. Quadratic vertex kernel for split vertex deletion. *Theoretical Computer Science*, 833:164–172, 2020.

2 Akanksha Agrawal, Sudeshna Kolay, Daniel Lokshtanov, and Saket Saurabh. A faster FPT algorithm and a smaller kernel for block graph vertex deletion. In *Theoretical Informatics - 12th Latin American Symposium (LATIN)*, pages 1–13, 2016.

3 Akanksha Agrawal, Daniel Lokshtanov, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. Feedback vertex set inspired kernel for chordal vertex deletion. *ACM Trans. Algorithms*, 15(1), December 2018. `doi:10.1145/3284356`.

4 Akanksha Agrawal, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. Interval vertex deletion admits a polynomial kernel. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '19, page 1711–1730, USA, 2019. Society for Industrial and Applied Mathematics.

5 Jungho Ahn. A polynomial kernel for 3-leaf power deletion. *Master's Thesis, KAIST, South Korea*, 2020.

6 Jungho Ahn, E. Eiben, O-joung Kwon, and Sang il Oum. A polynomial kernel for 3-leaf power deletion. In *MFCS*, 2020.

7 Jungho Ahn, Eun Jung Kim, and Euiwoong Lee. Towards constant-factor approximation for chordal/distance-hereditary vertex deletion. In *31st International Symposium on Algorithms and Computation (ISAAC 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

8 Yixin Cao and Dániel Marx. Interval deletion is fixed-parameter tractable. *ACM Transactions on Algorithms*, 11(3):21:1–21:35, 2015.

9 Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer-Verlag, 2015.

10 Marek Cygan and Marcin Pilipczuk. Split vertex deletion meets vertex cover: New fixed-parameter and exact exponential-time algorithms. *Information Processing Letters*, 113(5):179 – 182, 2013. URL: `http://www.sciencedirect.com/science/article/pii/S0020019013000069`, `doi:https://doi.org/10.1016/j.ipl.2013.01.001`.

11 Michael Dom, Jiong Guo, Falk Hüffner, and Rolf Niedermeier. Error compensation in leaf power problems. *Algorithmica*, 44:363–381, 04 2006. `doi:10.1007/s00453-005-1180-z`.

12 Jack Edmonds. Paths, trees and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.

13 Eduard Eiben, Robert Ganian, and O-joung Kwon. A single-exponential fixed-parameter algorithm for distance-hereditary vertex deletion. *Journal of Computer and System Sciences*, 97:121–146, 2018.

14 Martin Charles Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Elsevier, 2004.

15 Edward Howorka. A characterization of ptolemaic graphs. *Journal of Graph Theory*, 5(3):323–331, 1981.

16 Bart M. P. Jansen and Marcin Pilipczuk. Approximation and kernelization for chordal vertex deletion. *SIAM Journal on Discrete Mathematics*, 32(3):2258–2301, 2018.

17 Eun Jung Kim and O-joung Kwon. A polynomial kernel for distance-hereditary vertex deletion. In *Algorithms and Data Structures - 15th International Symposium (WADS)*, volume 10389, pages 509–520, 2017.

18 Jon M. Kleinberg and Éva Tardos. *Algorithm Design*. Addison-Wesley, 2006.

19 John M. Lewis and Mihalis Yannakakis. The node-deletion problem for hereditary properties is NP-complete. *Journal of Computer and System Sciences*, 20(2):219–230, 1980.

20 Dániel Marx. Chordal deletion is fixed-parameter tractable. *Algorithmica*, 57(4):747–768, 2010.

21 Ryuhei Uehara and Yushi Uno. Laminar structure of ptolemaic graphs with applications. *Discrete Appl. Math.*, 157(7):1533–1543, April 2009. `doi:10.1016/j.dam.2008.09.006`.

22 Mihalis Yannakakis. Node- and edge-deletion np-complete problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC)*, pages 253–264, 1978.