Interval Vertex Deletion Admits a Polynomial Kernel*

Akanksha Agrawal† Pranabendu Misra‡ Saket Saurabh§ Meirav Zehavi¶

Abstract

Given a graph G and an integer k, the Interval Vertex Deletion (IVD) problem asks whether there exists a subset $S \subseteq V(G)$ of size at most k such that $G - S$ is an interval graph. This problem is known to be NP-complete [Yannakakis, STOC'78]. Originally in 2012, Cao and Marx showed that IVD is fixed parameter tractable: they exhibited an algorithm with running time $10^k n^{\mathcal{O}(1)}$ [Cao and Marx, SODA'14]. The existence of a polynomial kernel for IVD remained a well-known open problem in Parameterized Complexity. In this paper, we settle this problem in the affirmative.

*The research leading to these results received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement no. 306992.

†Hungarian Academy of Sciences, Budapest, Hungary. akanksha@sztaki.hu.

‡University of Bergen, Bergen, Norway. pranabendu.misra@uib.no.

§University of Bergen, Bergen, Norway. The Institute of Mathematical Sciences, HBNI, Chennai, India. UMI ReLax. saket@imsc.res.in.

¶Ben-Gurion University, Beersheba, Israel. meiravze@bgu.ac.il.

1 Introduction

In a *graph modification problem*, the input consists of an n-vertex graph G and an integer k. The objective is to determine whether k *modification operations*—such as vertex deletions, or edge deletions, insertions or contractions—are sufficient to obtain a graph with prescribed structural properties such as being planar, bipartite, chordal, interval, acyclic or edgeless. Graph modification problems include some of the most basic problems in graph theory and graph algorithms. Unfortunately, most of these problems are NP-complete [43, 51]. Therefore, they have been studied intensively within algorithmic paradigms for coping with NP-completeness [21, 25, 46], including approximation algorithms, parameterized complexity, and algorithms for restricted input classes.

Graph modification problems have played a central role in the development of parameterized complexity, see the related works subsection. Here, the number of allowed modifications, k, is considered a *parameter*. With respect to k, we seek a *fixed parameter tractable (FPT)* algorithm, namely, an algorithm whose running time has the form $f(k)n^{\mathcal{O}(1)}$ for some computable function f. One way to obtain such an algorithm is to exhibit a *kernelization algorithm*, or *kernel*. A kernel for a graph problem Π is an algorithm that given an instance (G, k) of Π, runs in polynomial time and outputs an equivalent instance (G', k') of Π such that $|V(G')|$ and k' are upper bounded by $f(k)$ for some computable function f. The function f is called the *size* of the kernel, and if f is a polynomial function, then we say that the kernel is a *polynomial kernel*. A kernel for a problem immediately implies that it admits an FPT algorithm, but kernels are also interesting in their own right. In particular, kernels allow us to model the performance of polynomial time pre-processing algorithms. The field of kernelization has received a significant amount of attention, especially after the introduction of methods for showing kernelization lower bounds [5, 14, 15, 18, 24, 29, 30]. We refer to the surveys [23, 28, 39, 44], as well as the books [12, 17, 19, 49], for a detailed treatment of the area of kernelization. In this paper, we study the kernelization complexity of the following problem.

INTERVAL VERTEX DELETION (IVD)	**Parameter:** k
Input: A graph G and an integer k.	
Question: Does there exist a subset $S \subseteq V(G)$ of size at most k such that $G - S$ is an interval graph?	

A graph G is an *interval graph* if it is the intersection graph of intervals on the real line. Due to their intriguing combinatorial properties and many applications in diverse areas, such as industrial engineering and archeology [4, 36], the class of interval graphs is perhaps one of the most studied graph classes [7, 27]. Whether IVD admits an FPT algorithm has been a longstanding open problem in the area until it was resolved by Cao and Marx [10], who gave an algorithm with running time $\mathcal{O}(10^k n^9)$. Subsequently, Cao [9] designed an FPT algorithm with linear dependence on the input size, as well as slightly better dependence on the parameter k. More precisely, Cao's algorithm has running time $\mathcal{O}(8^k(n+m))$. A natural follow-up question to this work, explicitly asked multiple times in the literature [13, 31, 33], is whether IVD admits a polynomial kernel. In this paper, we resolve this question in the affirmative:

Theorem 1. INTERVAL VERTEX DELETION *admits a polynomial kernel.*

1.1 Methods

The first ingredient of our kernelization algorithm is the factor 8 polynomial time approximation algorithm for IVD by Cao [9]. We use this algorithm to obtain an approximate solution of size at most $8k$, or conclude that no solution of size at most k exists. By re-running the approximation algorithm on the graph with some of the vertices marked as "undeletable", we

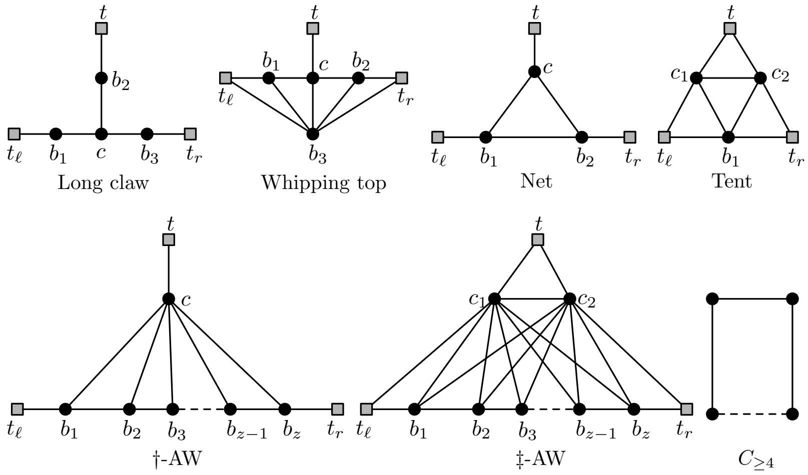

Figure 1: The set of obstructions for an interval graph.

grow our approximate solution to a 9-*redundant* solution M of size $\mathcal{O}(k^{10})$. Here, 9-redundancy roughly means that for every subset $W \subseteq M$ of size at most 9, either $M \setminus W$ is also a solution, or every solution S' of size at most $k+2$ has non-empty intersection with W.[1]

Our kernelization heavily uses the characterization of interval graphs in terms of their *forbidden induced subgraphs*, also called *obstructions*. Specifically, a graph H is an obstruction to the class of interval graphs if H is not an interval graph, and for every vertex $v \in V(H)$ we have that $H - \{v\}$ is an interval graph. A graph G is an interval graph if and only if it does not contain any obstruction as an induced subgraph. The set of obstructions to interval graphs have been completely characterized by Lekkerkerker and Boland, [42]. It consists of the *long claw*, the *whipping top*, the *net*, the *tent*, as well as three infinite families of graphs: the *single-dagger asteroidal witness* (†-AW), the *double-dagger asteroidal witnesses* (‡-AW), and the cycle of length at least 4 (see Figure 1).

Having a 9-redundant solution yields the following advantage. In several places, we remove a carefully chosen vertex $v \notin M$ from G and claim that $G - \{v\}$ has a solution of size at most k if and only if G does. One direction of the equivalence is trivial. The interesting direction is to show that a solution X of size k to $G - \{v\}$ implies the existence of a solution of size at most k for G. The starting point for such an analysis is to ask why X is not already a solution for G. The only possible reason is that $G - X$ contains an obstruction $\mathbb{O}$, and $\mathbb{O}$ must contain v. We claim that $\mathbb{O}$ contains at least 10 vertices from M. Suppose not, then let W be the intersection of M and $\mathbb{O}$. We know that $(G - (M \setminus W))$ contains $\mathbb{O}$, and therefore it is not an interval graph. Hence, by the 9-redundancy of M, this implies that X (being a solution of size at most $k+2$) must intersect $\mathbb{O}$, which contradicts the choice of $\mathbb{O}$. Thus, in this analysis we only need to care about *large* obstructions that, furthermore, have a large intersection with M. This is crucial throughout the design and analysis of the kernel.

We then proceed to classify the connected components of $G - M$ based on whether they are *modules* in G or not. (Recall that a module is a set X such that all vertices in X have the same neighbors outside X.) For each component C that is not a module, there is an edge (u, v) in C and a vertex w in M such that w is adjacent to u but not to v. Thus, if there are more than $(k+2)|M|$ non-module components in total, then there must exist $k+3$ non-module components and a vertex $w \in M$ such that each of these components has an edge (u, v) where w is adjacent to u but not to v. However, this means that for every subset $S \subseteq V(G)$ of size at most k, either

[1]The precise definition in Section 3 contains another condition that is not specified in the introduction for the sake of clarity of exposition.

$w \in S$ or $G - S$ contains a long claw (whose center c is w) and hence not interval. It follows that w must belong to every solution of size at most $k+2$; thus, we can simply remove w and decrease the budget k by 1. Hence, the number of non-module components can be bounded by $(k+2)|M|$, which is polynomial in k.

Since none of the obstructions contains any module on more than a single vertex, and the components of $G - M$ are interval graphs, it follows that every obstruction can intersect every module component in at most one vertex. Furthermore, there is no point in keeping more than $k+1$ copies of any vertex, so we can reduce the module components to cliques of size $k+1$.

We are left with the following situation. We have a 9-redundant solution M of size $\mathcal{O}(k^{10})$. At most $\mathcal{O}(|M|)$ components of $G - M$ are not modules, but these components could be arbitrarily large. The remaining components are all modules that are cliques of size at most $k+1$; thus, the module components are structured and small, but there could be arbitrarily many of them. This means that we are left with two tasks: *(i)* reduce the *number* of module components, and *(ii)* reduce the *size* of the non-module components. These two tasks can be approached separately, and both turn out to be non-trivial. Since both tasks are quite technically involved, we only give a few highlights in the remainder of this overview.

Bounding the Number of Module Components. Consider first the case where there are no non-module components at all, and every module component is a single vertex. In this case, $G - M$ is edgeless, so M is a *vertex cover* of G. The kernelization complexity of even this very special case was asked as an open problem by Fomin et al. [20].

A key ingredient in our solution to this special case is a new bound for the setting considered in the classic two families theorem of Bollobás [6]. Suppose there are two families of sets over a universe U, $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$, such that every set A_i has size p, every set B_j has size q, for every i the sets A_i and B_i are disjoint, while for every $i \neq j$ the sets A_i and B_j intersect. The two families theorem gives an upper bound of $\binom{p+q}{p}$ for the size m of the family. The upper bound on m is *independent of the universe size*, and this has been extensively used in the design of parameterized algorithms [22, 47]. Further, when p or q is a *constant* the bound is *polynomial* in $p+q$, and this has been extensively used in kernelization [40].

In our setting neither the sets $A_1, \ldots, A_m$ nor the sets $B_1, \ldots, B_m$ have constant cardinality. However, we know that for every $i \neq j$, $|A_i \cap B_j| \in \{1, 2\}$. We prove that in this case, the bound is $\mathcal{O}(|U|^2)$. More generally, we prove the following.

Lemma 1.1 (Bounded Intersection Two Families Lemma)**.** *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be families over a universe U such that (i) for every $i \leq m$, $A_i \cap B_i = \emptyset$, and (ii) for every $j \neq i$, $|A_i \cap B_j| \in \{1, \ldots, c\}$. Then $m \leq \sum_{t=0}^{c} \binom{|U|}{t}$.*

Comparing Lemma 1.1 with the Two Families Theorem, the bound in Lemma 1.1 does depend on the universe size $|U|$. On the other hand, the exponent of $|U|$ only depends on the maximum cardinality c of the *intersection* between the sets A_i and B_j.

In the setting of kernelizing IVD parameterized by the size of a vertex cover M, the size of the kernel is intimately linked to m for the case where $A_1, \ldots, A_m$ is a collection of cliques in $G[M]$ while $B_1, \ldots, B_m$ is a collection of induced paths. Since a clique can only intersect an induced path in at most two vertices, we can apply Lemma 1.1 with $c = 2$, thereby obtaining an $\mathcal{O}(|M|^2)$ bound for m and (after a significant amount of additional efforts, which we skip in this overview) a polynomial bound on the kernel size.

The kernel for IVD parameterized by vertex cover quite simply translates into a procedure that bounds the number, and therefore the total size, of module components of $G - M$. We remark that, because the *number* of non-module components is bounded by $\mathcal{O}(k|M|)$, by bounding the number of module components we also bound the total number of components of $G - M$.

Bounding the Size of Non-Module Components. Suppose now that the number of module components has been bounded by $k^{\mathcal{O}(1)}$. We can now include all of the module components in M, and proceed under the assumption that there are no module components at all.

The size-reduction of non-module components proceeds in three phases. In the first phase, we bound the maximum clique size in a component. Our clique-reduction procedure builds upon the clique-reduction procedure of Marx [48], which was used in kernelizations for Chordal Vertex Deletion [1, 34]. Both the procedure of Marx and ours are based on an "irrelevant vertex rule". However, our procedure is necessarily much more involved—our irrelevant vertex rule needs to preserve not only long induced cycles, but also large single and double dagger asteroidal witnesses.

Having reduced the maximum clique size in the component we proceed to the second phase, where we reduce the set of vertices that appear in at least two maximal cliques in the component. In this phase, we partition the component into $k^{O(1)}$ "long" and "thin" parts, and prove that an optimal solution will either not touch a part at all, or it will cut it into two pieces using a minimal separator. Then, provided that a part is sufficiently large, we identify an edge whose contraction does not decrease the size of any minimal separator inside the part. Thus, on the one hand, contracting e does not decrease the size of an optimal solution. On the other hand, contracting e—or any edge for that matter—cannot *increase* the size of an optimal solution (since interval graphs are closed under contraction).

After the second phase, the number of vertices appearing in at least two maximal cliques of the component is upper bounded by $k^{\mathcal{O}(1)}$. In the third phase, we bound the number of the remaining vertices—these are the vertices that are "private" to some maximal clique of the component. At this point we can take the set of vertices appearing in at least two components and add them to M. This makes M grow by $k^{\mathcal{O}(1)}$ vertices, but now the large component breaks up into components whose size is not larger than that of a maximal clique, that is, $k^{\mathcal{O}(1)}$. We can now re-apply the procedure for bounding the number of components and this bounds the total number of vertices in G by $k^{\mathcal{O}(1)}$. *We remark that, for technical reasons, in the actual proof phases* 2 *and* 3 *as described here are interleaved.*

1.2 Related Works on Parameterized Graph Modification Problems

The $\mathcal{F}$-Vertex Deletion problems corresponding to the families of edgeless graphs, forests, chordal graphs, interval graphs, bipartite graphs, and planar graphs are known as Vertex Cover, Feedback Vertex Set, Chordal Vertex Deletion, Interval Vertex Deletion, Odd Cycle Transversal/Vertex Bipartization and Planar Vertex Deletion, respectively. These problems are among the most well studied problems in the field of parameterized complexity. The study of parameterized graph deletion problems together with their various restrictions and generalizations has been an extremely active subarea over the last few years. In fact, just over the course of the last few years there have been results on parameterized algorithms for Chordal Editing [11], Unit Vertex (Edge) Deletion [8, 35], Interval Vertex (Edge) Deletion [9, 10], Planar $\mathcal{F}$ Deletion [21, 38], Planar Vertex Deletion [32], Block Graph Deletion [37] and Simultaneous Feedback Vertex Set [3]. It is important to note that for many of these problems, polynomial kernels gave rise to several new techniques in the area. However, the problem which is closest to ours is the Chordal Vertex Deletion problems. In a recent breakthrough, Jansen and Pilipczuk [34] gave a polynomial kernel (of size $\mathcal{O}(k^{162})$) for Chordal Vertex Deletion, resolving a more than a decade old open problem. Shortly afterwards, Agrawal et al. [1, 2] gave a kernel of size $\mathcal{O}(k^{13})$.

2 Preliminaries

We denote the set of natural numbers by $\mathbb{N}$. For $n \in \mathbb{N}$, we use $[n]$ and $[n]_0$ as shorthands for $\{1, 2, \ldots, n\}$ and $\{0, 1, \ldots, n\}$, respectively. For a set X and an integer $n \in \mathbb{N}$, by X^n we denote the set $\{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in X\}$.

Basic Graph Theory. We refer to standard terminology from the book of Diestel [16] for those graph-related terms that are not explicitly defined here. Given a graph G, we denote its vertex set and its edge set by $V(G)$ and $E(G)$, respectively. Given a set $\mathcal{C}$ of connected components of G, denote $V(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} V(C)$. Moreover, when the graph G is clear from context, denote $n = |V(G)|$. Given a subset $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. Accordingly, a graph H is an *induced subgraph* of G if there exists $U \subseteq V(G)$ such that $G[U] = H$. For a set of vertices $X \subseteq V(G)$, $G - X$ denotes the induced subgraph $G[V(G) \setminus X]$, i.e. the graph obtained by deleting the vertices in X from G. For an edge $(u, v) \in E(G)$, $G/(u, v)$ denotes the graph obtained by contracting the edge (u, v), i.e. the graph obtained by introducing a new vertex that is adjacent to all vertices in $N(u) \cup N(v)$ and deleting the vertices $\{u, v\}$. We say that G is a *clique* if for all distinct vertices $u, v \in V(G)$, we have that $(u, v) \in E(G)$, and that $V(G)$ is an independent set if for all distinct vertices $u, v \in V(G)$, we have that $(u, v) \notin E(G)$. Given a vertex $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v in G. Moreover, a subset $U \subseteq V(G)$ is a *module* if for all $u, u' \in U$ and $v \in V(G) \setminus U$, either both u and u' are adjacent to v or both u and u' are not adjacent to v. For the sake of simplicity, we also call $G[U]$ a module.

A *path* $P = (x_1, x_2, \ldots, x_\ell)$ in G is a subgraph of G where $V(P) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(P) = \{(x_i, x_{i+1}) \mid i \in [\ell - 1]\} \subseteq E(G)$, where $\ell \in [n]$. The vertices x_1 and x_ℓ are the *endpoints* of P, and the remaining vertices in $V(P)$ are the *internal vertices* of P. A *cycle* $C = (x_1, x_2, \ldots, x_\ell)$ in G is a subgraph of G where $V(C) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(C) = \{(x_i, x_{i+1}) \mid i \in [\ell - 1]\} \cup \{(x_1, x_\ell)\} \subseteq E(G)$. We say that $(u, v) \in E(G)$ is a *chord* of a path P if $u, v \in V(P)$ but $(u, v) \notin E(P)$. Similarly, we say that $(u, v) \in E(G)$ is a *chord* of a cycle C if $u, v \in V(C)$ but $(u, v) \notin E(C)$. A path P or cycle C is said to be *induced* (or, alternatively, *chordless*) if it has no chords.

Interval Graphs. An *interval graph* is a graph that does not contain any of the following graphs, called *obstructions*, as an induced subgraph (see Figure 1).

- **Long Claw.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{(t_\ell, b_1), (t_r, b_3), (t, b_2), (c, b_1), (c, b_2), (c, b_3)\}$.

- **Whipping Top.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{(t_\ell, b_1), (t_r, b_2), (c, t), (c, b_1), (c, b_2), (b_3, t_\ell), (b_3, b_1), (b_3, c), (b_3, b_2), (b_3, t_r)\}$.

- **†-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{(t, c), (t_\ell, b_1), (t_r, b_z)\} \cup \{(c, b_i) \mid i \in [z]\} \cup \{(b_i, b_{i+1}) \mid i \in [z - 1]\}$, and $z \geq 2$. A †-AW where $z = 2$ will be called a *net*.

- **‡-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c_1, c_2\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{(t, c_1), (t, c_2), (c_1, c_2), (t_\ell, b_1), (t_r, b_z), (t_\ell, c_1), (t_r, c_2)\} \cup \{(c, b_i) \mid i \in [z]\} \cup \{(b_i, b_{i+1}) \mid i \in [z - 1]\}$, and $z \geq 1$. A ‡-AW where $z = 1$ will be called a *tent*.

- **Hole.** A chordless cycle on at least four vertices.

An obstruction $\mathbb{O}$ is *minimal* if there does not exist an obstruction $\mathbb{O}'$ such that $V(\mathbb{O}') \subset V(\mathbb{O})$. We refer to †-AW and ‡-AW as AWs. In each of the first four obstructions, the vertices t_ℓ, t_r, and t are called *terminals*, the vertices c, c_1, and c_2 are called *centers*, and the other vertices are called *base vertices*. Furthermore, the vertex t is called the *shallow terminal* and the vertices t_ℓ and t_r are called the *non-shallow terminals*. In the case where $\mathbb{O}$ is one of the AWs, the induced path on the set of base vertices is called the *base* of the AW, and it is denoted by $\mathsf{base}(\mathbb{O})$. Moreover, we say that the induced path on the set of base vertices, t_ℓ and t_r is the *extended base* of the AW, and it is denoted by $P(\mathbb{O})$.

Path Decomposition. A *path decomposition* of a connected graph G is a pair (P, β) where P is a path, and $\beta : V(P) \to 2^{V(G)}$ is a function that satisfies the following properties.

(i) $\bigcup_{x \in V(P)} \beta(x) = V(G)$,

(ii) For any edge $(u, v) \in E(G)$ there is a node $x \in V(P)$ such that $u, v \in \beta(x)$.

(iii) For any $v \in V(G)$, the collection of nodes $P_v = \{x \in V(P) \mid v \in \beta(x)\}$ is a subpath of P.

For $v \in V(P)$, we call $\beta(v)$ the *bag* of v. We refer to the vertices in $V(P)$ as nodes. A *clique path* of a connected graph G is a path decomposition of G where every bag is a distinct maximal clique. If a graph G admits a clique path, then we say that G is a clique path. The following proposition states that the class of interval graphs is exactly the class of graphs where each connected component is a clique path.

Proposition 1 ([26, 27]). *A graph is an interval graph if and only if each connected component of it is a clique path.*

Parameterized Complexity. Let Π be an NP-hard problem. In the framework of Parameterized Complexity, each instance of Π is associated with an integer k, which is called the *parameter*. Here, the goal is to confine the combinatorial explosion in the running time of an algorithm for Π to depend only on k. The main concepts defined to achieve this goal are of *fixed-parameter tractability* and *kernelization*. First, we say that Π is *fixed-parameter tractable (FPT)* if any instance (I, k) of Π is solvable in time $f(k) \cdot |I|^{\mathcal{O}(1)}$, where $f(\cdot)$ is an arbitrary (computable) function of k. Second, Π is said to admit a *polynomial kernel* if there is a polynomial-time algorithm (the degree of polynomial is independent of the parameter k), called a *kernelization algorithm*, that transforms the input instance into an equivalent instance of Π whose size is bounded by a polynomial $p(k)$ in k. Here, two instances are equivalent if one of them is a Yes-instance if and only if the other one is a Yes-instance. The reduced instance is called a $p(k)$*-kernel* for Π. For a detailed introduction to the field of kernelization, we refer to the following surveys [39, 44] and the corresponding chapters in the books [12, 17, 19, 49].

Kernelization algorithms often rely on the design of *reduction rules*. The rules are numbered, and each rule consists of a condition and an action. We always apply the first rule whose condition is true. Given a problem instance (I, k), the rule computes (in polynomial time) an instance (I', k') of the same problem where $k' \leq k$. Typically, $|I'| < |I|$, where if this is not the case, it should be argued why the rule can be applied only polynomially many times. We say that the rule *safe* if the instances (I, k) and (I', k') are equivalent.

Linear Algebra. For a set A and X, by an *operation of A onto X* we mean a function $f : A \times X \to X$. For an element $(a, x) \in A \times X$ by ax we denote the element $f(a, x) \in X$. For a field $\mathbb{F}$ with $+$ as the additive operation and $\cdot$ as the multiplicative operation, a commutative group $(V, +)$ with an operation of $\mathbb{F}$ onto V is a *vector space over* $\mathbb{F}$ if for all $a, b \in \mathbb{F}$ and $x, y \in V$, we have: **i)** $a(bx) = (ab)x$; **ii)** $a(x + y) = ax + ay$; **iii)** $(a + b)x = ax + bx$; **iv)** $1 \cdot x = x$. Here, 1 is the additive identity of the field $\mathbb{F}$. If V is a vector space over $\mathbb{F}$, then the elements of V are called vectors. One of the natural candidates for vector spaces over a field $\mathbb{F}$ is $\mathbb{F}^n$, where $n \in \mathbb{N}$ and the function $f(\cdot)$ is the component-wise multiplication. In this paper, we restrict ourselves only to such types of vector spaces.

In the following, consider a field $\mathbb{F}$ and a vector space $V = \mathbb{F}^n$, where $n \in \mathbb{N}$. For a vector $\mathbf{v} = (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$ and an integer $i \in [n]$, by $\mathbf{v}[i]$ we denote the ith element (or entry) of

$\mathbf{v}$, i.e., the element b_i. For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t \in \mathbb{F}^n$, a linear combination of them is a vector $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_t\mathbf{v}_t$, where $a_1, a_2, \ldots, a_t \in \mathbb{F}$. Furthermore, a *linear relation* among them is exhibited when $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_t\mathbf{v}_t = 0$, for some $a_1, a_2, \ldots, a_t \in \mathbb{F}$. In the above, the a_is are called the *coefficients*. A set of vectors is said to be *linearly independent* if there is no linear relation among them except the trivial one, where each of the coefficients is 0. A set of vectors that is not linearly independent is said to be *linearly dependent*. An inclusion-wise maximal set of linearly independent vectors is called a *basis* of the vector space. It is known that for bases B, B' of a vector space, we have $|B| = |B'|$. By $\mathbb{F}_2$ we denote the field with exactly two elements, namely 0 and 1, with the usual addition and multiplication modulo 2 as the field operations. For two vectors $\mathbf{u}, \mathbf{v} \in V'$, $\mathbf{u} \cdot \mathbf{v}$ denotes the dot product of these two vectors. We refer the reader to [41] for more details on linear algebra.

Matroids. A pair $\mathcal{M} = (E, \mathcal{I})$, where E is a set (called *ground set*) and $\mathcal{I}$ is a family of subsets of E (called *independent sets*) is called a *matroid* if the following conditions are satisfied.

- $\emptyset \in \mathcal{I}$,
- If $A \in \mathcal{I}$ and $A' \subseteq A$ then $A' \in \mathcal{I}$,
- If $A, B \in \mathcal{I}$ and $|A| < |B|$ then there is $x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{I}$.

An inclusion-wise maximal set in $\mathcal{I}$ is called a *basis* of $\mathcal{M}$. All the basis of a matroid are of same size. The size of a basis is called the rank of the matroid. One of the important notions of a matroid which we use is linear representations of matroids.

A matroid is a *linear matroid* (or *representable matroid*) if, for some field $\mathbb{F}$, it can be defined as follows. Let A be a matrix over a field $\mathbb{F}$ and E its set of columns. Then, the matroid $\mathcal{M} = (E, \mathcal{I})$ is defined as follows: a subset $X \subseteq E$ is an independent set in $\mathcal{M}$ if and only if the set of columns in X is linearly independent over $\mathbb{F}$. The matrix A is called a *representation* of $\mathcal{M}$, and $\mathcal{M}$ is said to be *representable over* $\mathbb{F}$. Thus, matroid is linear (alternatively, representable) if it is representable over some field $\mathbb{F}$. We refer the reader to [50] for more details on matroids.

q-Representative Family. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $\mathcal{B}$ be a family of subsets of size p of E. We say that $\widehat{\mathcal{B}} \subseteq \mathcal{B}$ is a *q-representative for* $\mathcal{B}$ if for every set $Y \subseteq E$ of size q: if there is a set $X \in \mathcal{B}$ such that $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$, then there is a set $\widehat{X} \in \widehat{\mathcal{B}}$ such that $\widehat{X} \cap Y = \emptyset$ and $\widehat{X} \cup Y \in \mathcal{I}$. If $\widehat{\mathcal{B}} \subseteq \mathcal{B}$ is a q-representative for $\mathcal{B}$, then we use the notation $\widehat{\mathcal{B}} \subseteq^q_{rep} \mathcal{B}$. The following result asserts that small representative families can be computed efficiently.

Proposition 2 ([22])**.** *Let $\mathcal{M} = (E, \mathcal{I})$ be a linear matroid of rank $k = p+q$, and matrix $A_{\mathcal{M}}$ be a representation of $\mathcal{M}$ over a field $\mathbb{F}$. Also, let $\mathcal{B} = \{B_1, B_2, \ldots, B_t\}$ be a family of independent sets of size p over E. Then, there exists $\widehat{\mathcal{B}} \subseteq^q_{rep} \mathcal{B}$ of size at most $\binom{p+q}{p}$. Moreover, such $\widehat{\mathcal{B}}$ can be computed in at most $\mathcal{O}(\binom{p+q}{p}tp^\omega + t\binom{p+q}{p}^{\omega-1})$ operations over $\mathbb{F}$. Here, ω is the exponent in the running time of matrix multiplication.*

3 Computing a Redundant Solution

Let (G, k) be an instance of IVD. A subset $S \subseteq V(G)$ such that $G - S$ is an interval graph is called a *solution*, and a solution of size at most t is called a *t-solution*. Towards the definition of redundancy, we need to introduce a few simple notions related to hitting and covering. Given a family $\mathcal{W} \subseteq 2^{V(G)}$, we say that a subset $S \subseteq V(G)$ *hits* $\mathcal{W}$ if for all $W \in \mathcal{W}$, we have $S \cap W \neq \emptyset$. A family $\mathcal{W} \subseteq 2^{V(G)}$ is *t-necessary* if every solution of size at most t hits $\mathcal{W}$. Moreover, we say that an obstruction $\mathbb{O}$ is *covered by* $\mathcal{W}$ if there exists $W \in \mathcal{W}$, such that $W \subseteq V(\mathbb{O})$. Now, we are ready to formally define our notion of redundancy.

Definition 3.1. Given a family $\mathcal{W} \subseteq 2^{V(G)}$ and $t \in \mathbb{N}$, a subset $M \subseteq V(G)$ is *t-redundant with respect to $\mathcal{W}$* if for every obstruction $\mathbb{O}$ that is not covered by $\mathcal{W}$, it holds that $|M \cap V(\mathbb{O})| > t$.

The purpose of this section is to prove Lemma 3.1 below. Intuitively, this lemma asserts that an r-redundant solution M whose size is polynomial in k (for a fixed constant r) can be computed in polynomial time. Such a set M plays a central role in all of our subsequent reduction rules that comprise our kernelization algorithm. We remark that in this statement we use the letter ℓ rather than k to avoid confusion, as we will use this result with $\ell = k + 2$.

Lemma 3.1. *Let $r \in \mathbb{N}$ be a fixed constant, and (G, ℓ) be an instance of* IVD*. In polynomial time, it is possible to either conclude that (G, ℓ) is a* No*-instance, or compute an ℓ-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ and a set $M \subseteq V(G)$, such that $\mathcal{W} \subseteq 2^M$ and M is a $(r + 1)(6\ell)^{r+1}$-solution that is r-redundant with respect to $\mathcal{W}$.*

A central component in our proof of Lemma 3.1 is an approximation algorithm for IVD, given by Cao [9]:

Proposition 3 ([9])**.** IVD *admits a polynomial-time 6-approximation algorithm, called* `ApproxIVD`*.*

In particular, a main idea in our proof is to iteratively grow the redundancy of a solution by making calls to this approximation algorithm. Besides Proposition 3, towards the proof of Lemma 3.1, we give a simple definition of a graph on which we will apply the approximation algorithm and hence determine whether a set of vertices should be added to $\mathcal{W}$.

Definition 3.2. Let G be a graph, $U \subseteq V(G)$, and $t \in \mathbb{N}$. Then, $\mathsf{copy}(G, U, t)$ is defined as the graph G' on the vertex set $V(G) \cup \{v^i \mid v \in U, i \in [t]\}$ and the edge set $E(G) \cup \{(u^i, v) \mid (u, v) \in E(G), u \in U, i \in [t]\} \cup \{(u^i, v^j) \mid (u, v) \in E(G), u, v \in U, i, j \in [t]\} \cup \{(v, v^i) \mid v \in U, i \in [t]\} \cup \{(v^i, v^j) \mid v \in U, i, j \in [t], i \neq j\}$.

Informally, $\mathsf{copy}(G, U, t)$ is simply the graph G where for every vertex $u \in U$, we add t twins that (together with u) form a clique. Intuitively, this operation allows us to make a vertex set "undeletable"; in particular, this enables us to test later whether a vertex set is "redundant" and hence we can grow the redundancy of our solution, or whether it is "necessary" and hence we should update $\mathcal{W}$ accordingly. Before we turn to discuss computational issues, let us first assert that the operation in Definition 3.2 does not makes an interval graph become a non-interval graph. This is a basic requirement to verify before turning to design the above mentioned test.

Lemma 3.2. *Let G be a graph, $U \subseteq V(G)$, and $t \in \mathbb{N}$. If G is an interval graph, then $G' = \mathsf{copy}(G, U, t)$ is an interval graph as well.*

Proof. Suppose that G is an interval graph. Then, by Proposition 1, G admits a clique path (P, β). Now, we define (P', β') as follows: $P' = P$, and for all $x \in V(P')$, $\beta'(x) = \beta(x) \cup \{v^i \mid v \in \beta(x) \cap U, i \in [t]\}$. We claim that (P', β') is a clique path for G'. By using the fact that (P, β) is a path decomposition of G, we directly have the following properties. First, it is clear that $\bigcup_{x \in V(P')} \beta'(x) = V(G')$. Second, for any edge $e = (u, v) \in E(G')$ such that $u, v \in V(G)$, there exists $x_e \in V(P')$ such that $u, v \in \beta'(x_e)$. Then, since for all $v \in U$ and $i \in [t]$, it holds that $\beta'^{-1}(v) = \beta'^{-1}(v^i)$, we derive that for any edge $(u', v') \in E(G')$ there is a node $x \in V(P')$ such that $u', v' \in \beta'(x)$. Third, for any $v \in V(G)$, the collection of nodes $P'_v = \{x \in V(P') \mid v \in \beta'(u)\}$ is a subpath of P', and since for any $v \in U$ and $i \in [t]$, it holds that $\beta'^{-1}(v) = \beta'^{-1}(v^i)$, we derive that for any $v' \in V(G')$, the collection of nodes $P'_{v'} = \{x \in V(P') \mid v' \in \beta'(x)\}$ is a subpath of P'. Now, note that for all $x \in V(P')$, $\beta(x)$ is a clique, and for all $u, v \in \beta(x)$ (possibly $u = v$) and $i, j \in [t]$, u^i is adjacent to u, u^j (if $i \neq j$), v and v^j, which implies that $\beta'(x)$ is also a clique. Hence, (P', β') is indeed clique path for G'. By Proposition 1, we derive that G' is an interval graph. $\square$

Now, let us present two simple claims that exhibit relations between the algorithm `ApproxIVD` and Definition 3.2. After presenting these two claims, we will be ready to give our algorithm for computing a redundant solution. Roughly speaking, the first claim exhibits the meaning of a situation where `ApproxIVD` returns a "large" solution; intuitively, for the purpose of the design of our algorithm, we interpret this meaning as an indicator to extend $\mathcal{W}$.

Lemma 3.3. *Let G be a graph, $U \subseteq V(G)$, and $\ell \in \mathbb{N}$. If the algorithm ApproxIVD returns a set A of size larger than 6ℓ when called with $G' = \mathsf{copy}(G, U, 6\ell+1)$ as input, then $\{U\}$ is ℓ-necessary.*

Proof. Suppose that `ApproxIVD` returns a set A of size larger than 6ℓ when called with G' as input. Then, (G', ℓ) is a No-instance. Suppose, by way of contradiction, that $\{U\}$ is not ℓ-necessary. Then, G has an ℓ-solution S such that $S \cap U = \emptyset$. In particular, $\widehat{G} = G - S$ is an interval graph such that $U \subseteq V(\widehat{G})$. However, this means that $\mathsf{copy}(\widehat{G}, U, 6\ell+1) = G' - S$, which by Lemma 3.2 implies that $G' - S$ is an interval graph. Thus, S is an ℓ-solution for G', which is a contradiction (as (G', ℓ) is a No-instance). □

Complementing our first claim, the second claim exhibits the meaning of a situation where `ApproxIVD` returns a "small" solution A; we interpret this meaning as an indicator to grow the redundancy of our current solution M by adding A—indeed, this lemma implies that every obstruction is hit one more time when adding A to a subset $U \subseteq M$ (to grow the redundancy of M, every subset $U \subseteq M$ will have to be considered).

Lemma 3.4. *Let G be a graph, $U \subseteq V(G)$, and $\ell \in \mathbb{N}$. If the algorithm ApproxIVD returns a set A of size at most 6ℓ when called with $G' = \mathsf{copy}(G, U, 6\ell+1)$ as input, then for every obstruction $\mathbb{O}$ of G, $|V(\mathbb{O}) \cap U| + 1 \leq |V(\mathbb{O}) \cap (U \cup (A \cap V(G)))|$.*

Proof. Suppose that `ApproxIVD` returned a set A of size at most 6ℓ when called with G' as input. Let $\mathbb{O}$ be some obstruction of G, and denote $B = V(\mathbb{O}) \cap U$. Since $|A| \leq 6\ell$, for every vertex $v \in B$, we have that $v \in V(G') \setminus A$ or there exists $i(v) = i \in [6\ell]$ such that $v^i \in V(G') \setminus A$. Moreover, we have that the graph obtained from $\mathbb{O}$ by replacing each vertex $v \in B \cap A$ by $v^{i(v)}$ is an obstruction (as v and $v^{i(v)}$ are twins). Thus, as A is a solution for G', there exists $v \in V(G) \setminus B$ such that $v \in A \cap V(\mathbb{O})$. Hence, we have that $|V(\mathbb{O}) \cap U| + 1 \leq |V(\mathbb{O}) \cap (U \cup (A \cap V(G)))|$. □

Now, let us describe our algorithm, `RedundantIVD`, to compute a redundant solution. First, `RedundantIVD` initializes M_0 to be the output obtained by calling the algorithm `ApproxIVD` with G as input, $\mathcal{W}_0 := \emptyset$ and $\mathcal{T}_0 := \{(v) \mid v \in M_0\}$. If $|M_0| > 6\ell$, then `RedundantIVD` concludes that (G, ℓ) is a No-instance. Otherwise, for $i = 1, 2, \ldots, r$ (in this order), the algorithm executes the following steps:

1. Initialize $M_i := M_{i-1}$, $\mathcal{W}_i := \mathcal{W}_{i-1}$ and $\mathcal{T}_i := \emptyset$.
2. For every tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$:
   (a) Let A be the output obtained by calling the algorithm `ApproxIVD` with $\mathsf{copy}(G, \{v_0, v_1, \ldots, v_{i-1}\}, 6\ell+1)$ as input.
   (b) If $|A| > 6\ell$, then insert $\{v_0, v_1, \ldots, v_{i-1}\}$ into $\mathcal{W}_i$.
   (c) Otherwise, insert every vertex in $(A \cap V(G)) \setminus \{v_0, v_1, \ldots, v_{i-1}\}$ into M_i, and for all $u \in (A \cap V(G)) \setminus \{v_0, v_1, \ldots, v_{i-1}\}$, insert $(v_0, v_1, \ldots, v_{i-1}, u)$ into $\mathcal{T}_i$.

Eventually, the algorithm outputs the pair $(M_r, \mathcal{W}_r)$.

Let us comment that in this algorithm, we make use of the sets $\mathcal{T}_{i-1}$ rather than going over all subsets of size i of M_{i-1} in order to obtain a substantially better algorithm in terms of the size of the produced redundant solution.

The properties of the algorithm `RedundantIVD` that are relevant to us are summarized in the following lemma and observation, which are proved by induction and by making use of Lemmata 3.2, 3.3 and 3.4. Roughly speaking, we first assert that, unless (G, ℓ) is concluded to be a No-instance, we compute sets $\mathcal{W}_i$ that are ℓ-necessary as well as that the tuples in $\mathcal{T}_i$ "hit more vertices" of the obstructions in the input as i grows larger.

Lemma 3.5. *Consider a call to* `RedundantIVD` *with (G, ℓ, r) as input that did not conclude that (G, ℓ) is a No-instance. For all $i \in [r]_0$, the following conditions hold:*

1. *For any set $W \in \mathcal{W}_i$, every solution S of size at most ℓ satisfies $W \cap S \neq \emptyset$.*
2. *For any obstruction $\mathbb{O}$ of G that is not covered by $\mathcal{W}_i$, there exists $(v_0, v_1, \ldots, v_i) \in \mathcal{T}_i$ such that $\{v_0, v_1, \ldots, v_i\} \subseteq V(\mathbb{O})$.*

Proof. The proof is by induction on i. In the base case, where $i = 0$, Condition 1 trivially holds as $\mathcal{W}_0 = \emptyset$, and Condition 2 holds as M_0 is a solution and $\mathcal{T}_0$ simply contains a 1-vertex tuple for every vertex in M_0. Now, suppose that the claim is true for $i - 1 \geq 0$, and let us prove it for i.

To prove Condition 1, consider some set $W \in \mathcal{W}_i$. If $W \in \mathcal{W}_{i-1}$, then by the inductive hypothesis, every solution of size at most ℓ satisfies $W \cap S \neq \emptyset$. Thus, we next suppose that $W \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}$. Then, there exists a tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$ in whose iteration `RedundantIVD` inserted $W = \{v_0, v_1, \ldots, v_{i-1}\}$ into $\mathcal{W}_i$. In that iteration, `ApproxIVD` was called with $\mathsf{copy}(G, W, 6\ell + 1)$ as input, and returned a set A of size larger than 6ℓ. Thus, by Lemma 3.3, every solution S of size at most ℓ satisfies $W \cap S \neq \emptyset$.

To prove Condition 2, consider some obstruction $\mathbb{O}$ of G that is not covered by $\mathcal{W}_i$. By the inductive hypothesis and since $\mathcal{W}_{i-1} \subseteq \mathcal{W}_i$, there exists a tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$ such that $\{v_0, v_1, \ldots, v_{i-1}\} \subseteq V(\mathbb{O})$. Consider the iteration of `RedundantIVD` corresponding to this tuple, and denote $U = \{v_0, v_1, \ldots, v_{i-1}\}$. In that iteration, `ApproxIVD` was called with $\mathsf{copy}(G, U, 6\ell + 1)$ as input, and returned a set A of size at most 6ℓ. By Lemma 3.4, $|V(\mathbb{O}) \cap U| + 1 \leq |V(\mathbb{O}) \cap (U \cup (A \cap V(G)))|$. Thus, there exists $v_i \in (A \cap V(G)) \setminus U$ such that $U \cup \{v_i\} \subseteq V(\mathbb{O})$. However, by the specification of `ApproxIVD`, this means that there exists $(v_0, v_1, \ldots, v_i) \in \mathcal{T}_i$ such that $\{v_0, v_1, \ldots, v_i\} \subseteq V(\mathbb{O})$. $\square$

Towards showing that the output set M_r is "small", let us upper bound the sizes of the sets M_i and $\mathcal{T}_i$.

Observation 3.6. *Consider a call to* `RedundantIVD` *with (G, ℓ, r) as input that did not conclude that (G, ℓ) is a No-instance. For all $i \in [r]_0$, $|M_i| \leq \sum_{j=0}^{i} (6\ell)^{j+1}$, $|\mathcal{T}_i| \leq (6\ell)^{i+1}$ and every tuple in $\mathcal{T}_i$ consists of distinct vertices.*

Proof. The proof is by induction on i. In the base case, where $i = 0$, the correctness follows as `ApproxIVD` returned a set of size at most 6ℓ. Now, suppose that the claim is true for $i - 1 \geq 0$, and let us prove it for i. By the specification of the algorithm and inductive hypothesis, we have that $|M_i| \leq |M_{i-1}| + 6\ell|\mathcal{T}_{i-1}| \leq \sum_{j=1}^{i+1} (6\ell)^j$ and $|\mathcal{T}_i| \leq 6\ell|\mathcal{T}_{i-1}| \leq (6\ell)^{i+1}$. Moreover, by the inductive hypothesis, for every tuple in $\mathcal{T}_i$, the first i vertices are distinct, and by the specification of `ApproxIVD`, the last vertex is not equal to any of them. $\square$

By the specification of `RedundantIVD`, as a corollary to Lemma 3.5 and Observation 3.6, we directly obtain the following result.

Corollary 3.7. *Consider a call to* `RedundantIVD` *with (G, ℓ, r) as input that did not conclude that (G, ℓ) is a No-instance. For all $i \in [r]_0$, $\mathcal{W}_i$ is an ℓ-necessary and M_i is a $\sum_{j=0}^{i} (6\ell)^{j+1}$-solution that is i-redundant with respect to $\mathcal{W}_i$.*

Clearly, `RedundantIVD` runs in polynomial time (as r is a fixed constant), and by the correctness of `ApproxIVD`, if it concludes that (G, ℓ) is a No-instance, then this decision is correct. Thus, since $\sum_{i=0}^{r}(6\ell)^{r+1} \leq (r+1)(6\ell)^{r+1}$, the correctness of Lemma 3.1 now directly follows as a special case of Corollary 3.7. Thus, our proof of Lemma 3.1 is complete.

In light of Lemma 3.1, from now on, we suppose that we have a $(k+2)$-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ along with a $(r+1)(6(k+2))^{r+1}$-solution M that is r-redundant with respect to $\mathcal{W}$ for $r = 9$. Let us note that, any obstruction in G that is not covered by $\mathcal{W}$ intersects M in at least ten vertices. We have the following reduction rule that follows immediately from Lemma 3.5.

Reduction Rule 3.1. *Let v be a vertex such that $\{v\} \in \mathcal{W}$. Then, output the instance $(G - \{v\}, k-1)$.*

Henceforward, we will assume that each set in $\mathcal{W}$ has size at least 2.

4 Handling Module Components

Let (G, k) be an instance of IVD. Let us explicitly recap the steps taken so far, and then state our current objective in this context. First, we call Lemma 3.1 with $r = 9$ and $\ell = k + 2$, and one of the following holds. If (in polynomial time) we conclude that $(G, k+2)$ is a No-instance, then we can (correctly) conclude that (G, k) is a No-instance as well. Otherwise, in polynomial time we obtain a $(k+2)$-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ and a set $M \subseteq V(G)$, such that $\mathcal{W} \subseteq 2^M$ and M is a $10(6(k+2))^{10}$-solution that is 9-redundant with respect to $\mathcal{W}$. Furthermore, each set in $\mathcal{W}$ has size at least 2. The main goal of this section is to bound the total number of vertices across all module connected components of $G - M$. We remark that we will prove a slightly more general result, as it will be used later in our algorithm. Before that, we provide a simple reduction rule to bound the number of non-module components.

Bounding the Number of Non-Module Components. Let $\mathcal{C}$ denote the set of connected components of $G - M$. Moreover, we let $\mathcal{D}$ denote the set of connected components in $\mathcal{C}$ that are modules, and $\overline{\mathcal{D}} = \mathcal{C} \setminus \mathcal{D}$. To bound the size of $\overline{\mathcal{D}}$, we apply the following reduction rule.

Reduction Rule 4.1. *Suppose that there exist $v \in M$ and a set $\mathcal{A} \subseteq \overline{\mathcal{D}}$ of size $k+3$ such that for each $D \in \mathcal{A}$, there exist $u, w \in V(D)$ such that $u \in N_G(v)$ and $w \notin N_G(v)$. Then, output the instance $(G - \{v\}, k-1)$.*

Lemma 4.1. *Reduction Rule 4.1 is safe.*

Proof. In one direction, suppose that (G, k) is a Yes-instance, and let S be a k-solution for G. Since $|\mathcal{A}| \geq k+3$, there exist three connected components $D_1, D_2, D_2 \in \overline{\mathcal{D}} \cap \mathcal{A}$ such that $S \cap (V(D_1) \cup V(D_2) \cup V(D_3)) = \emptyset$. However, for each $i \in [3]$, the subgraph of G induced by the vertex set consisting of v, together with an edge e in D_i with one endpoint of e being a neighbor of v and the other endpoint of e being a non-neighbor of v, is a long claw. Here, we relied on the fact that for each $i \in [3]$, D_i is connected. Thus, as $G - S$ is an interval graph, we derive that $v \in S$, and therefore $S \setminus \{v\}$ is a $(k-1)$-solution for $G - \{v\}$.

In the other direction, it is clear that if $(G - \{v\}, k-1)$ is a Yes-instance, then (G, k) is a Yes-instance. ☐

We now observe that our rule indeed bounds the size of $\overline{\mathcal{D}}$.

Observation 4.2. *After the exhaustive application of Reduction Rule 4.1, $|\overline{\mathcal{D}}| \leq (k+2)|M|$.*

Proof. After the exhaustive application of Reduction Rule 4.1, every vertex in M has at most $k+2$ connected components in $\mathcal{C}$ where it has both a neighbor and a non-neighbor. Since for a connected component in $\overline{\mathcal{D}}$ that is not a module, there must exist a vertex in M that has both a neighbor and a non-neighbor in that component, we conclude that the observation is correct. $\square$

The Main Lemma of this Section. From now on, we focus on the main goal of this section: bound the total number of vertices in $\mathcal{D}$. As mentioned earlier, the arguments used to derive this bound will also be necessary at a later stage of our kernelization algorithm, and hence we present our goal in the form of a more general statement:

Lemma 4.3. *Let $\widehat{M} \subseteq V(G)$, and $\widehat{\mathcal{C}}$ be some set of connected components of $G - (M \cup \widehat{M})$ that are modules. In polynomial time, it is possible to either output an instance (G', k) equivalent to (G, k) where G' is a strict subgraph of G, or to compute a subset $B \subseteq V(\widehat{\mathcal{C}})$ of size at most $4(k+1)^2|M \cup \widehat{M}|^6$, such that for any subset $S \subseteq V(G)$ of size at most k, the following property holds: If there exists an obstruction $\mathbb{O}$ for G that is not covered by $\mathcal{W}$ and such that $V(\mathbb{O}) \cap S = \emptyset$, there exists an obstruction $\mathbb{O}'$ for G such that $V(\mathbb{O}') \cap S = \emptyset$ and $V(\mathbb{O}') \cap (V(\widehat{\mathcal{C}}) \setminus B) = \emptyset$.*

Intuitively, the statement of this lemma expands M to $M \cup \widehat{M}$, and zooms into a subset $\widehat{\mathcal{C}}$ of the set of connected components that are modules in $G - (M \cup \widehat{M})$. Then, either it enables us to reduce the instance, or it produces a "small" subset $B \subseteq V(\widehat{\mathcal{C}})$ and implies that we need not "worry" about obstructions that intersect $V(\widehat{\mathcal{C}})$ but not B—if such an obstruction is not hit, then there is an obstruction that does not intersect $V(\widehat{\mathcal{C}}) \setminus B$ and which is not hit as well.

Let us now show that having Lemma 4.3 at hand, we can indeed bound the total number of vertices in all module components.

Reduction Rule 4.2. *Let X be the output of the algorithm in Lemma 4.3 when called with $\widehat{M} = \emptyset$ and $\widehat{\mathcal{C}} = \mathcal{D}$. If X is an instance (G', k), then output X. Otherwise, X is a set $B \subseteq V(\mathcal{D})$, and we output the instance $(G - \{v\}, k)$ for a vertex v arbitrarily chosen from $V(\mathcal{D}) \setminus B$.*

By using Lemma 4.3, we derive the safeness of Reduction Rule 4.2.

Lemma 4.4. *Reduction Rule 4.3 is safe.*

Proof. If X is an instance (G', k), then Lemma 4.3 directly implies that the rule is safe. Thus, we next suppose that $X = B$. In one direction, it is clear that if (G, k) is a Yes-instance, then $(G - \{v\}, k)$ is a Yes-instance as well.

In the other direction, suppose that $(G - \{v\}, k)$ is a Yes-instance. Let S be a k-solution for $G - \{v\}$. We claim that S is also a k-solution for G. Suppose, by way of contradiction, that this claim is false. Then, there exists an obstruction $\mathbb{O}$ for $G - S$. As $S \cup \{v\}$ is a $(k+1)$-solution for G and $\mathcal{W}$ is $(k+2)$-necessary, we have that $S \cup \{v\}$ hits $\mathcal{W}$. Since $v \notin M$ and $\mathcal{W} \subseteq 2^M$, we derive that S hits $\mathcal{W}$. Thus, since $\mathbb{O}$ is an obstruction for $G - S$, we deduce that $\mathbb{O}$ is not covered by $\mathcal{W}$. Hence, by Lemma 4.3, there exists an obstruction $\mathbb{O}'$ for G such that $V(\mathbb{O}') \cap S = \emptyset$ and $V(\mathbb{O}') \cap (V(\mathcal{D}) \setminus B) = \emptyset$. However, as $v \in V(\mathcal{D}) \setminus B$, this implies that $\mathbb{O}'$ is also an obstruction for $(G - \{v\}) - S$, which is a contradiction as S is a k-solution for $G - \{v\}$. $\square$

Due to Reduction Rule 4.2, we have the following result.

Observation 4.5. *After the exhaustive application of Reduction Rule 4.2, $|V(\mathcal{D})| \leq 4(k+1)^2|M|^6$.*

We now turn to prove Lemma 4.3. In what follows, $\widehat{M}$ and $\widehat{\mathcal{C}}$ are as stated in this lemma. We denote $M' = M \cup \widehat{M}$. Note that since M is 9-redundant with respect to $\mathcal{W}$, we have that M' is also 9-redundant with respect to $\mathcal{W}$. We begin our proof by showing that the common neighborhood outside M' of any two non-adjacent vertices, unless these two vertices form a pair in $\mathcal{W}$, is simply a clique. This simple claim will come in handy in several arguments later.

Lemma 4.6. *Let $u, v \in V(G)$ be distinct vertices such that $(u,v) \notin E(G)$ and $\{u,v\} \notin \mathcal{W}$. Then, $G[(N_G(u) \cap N_G(v)) \setminus M']$ is a clique.*

Proof. Suppose, by way of contradiction, that $G[N_G(u) \cap N_G(v) \setminus M']$ is not a clique. Then, there exist two vertices $x, y \in (N_G(u) \cap N_G(v)) \setminus M'$ that are not neighbors in G. Note that $\mathbb{O} = G[\{u,v,x,y\}]$ is a hole, and that $M \cap V(\mathbb{O}) \subseteq \{u,v\}$. Moreover, $\mathbb{O}$ is not covered by $\mathcal{W}$ (because $\{u,v\} \notin \mathcal{W}$ and every set in $\mathcal{W}$ has size at least 2). Since M is 9-redundant, this means that $|M \cap V(\mathbb{O})| > 9$. However, $|V(\mathbb{O})|$, hence we have reached a contradiction. $\square$

Structure of Obstructions Intersecting Module Components. In order to reduce our instance or to obtain a set B as required to prove Lemma 4.3, we need to understand how obstructions can intersect module components. For this purpose, we state a simple proposition by Cao and Marx [10]. This proposition asserts that because we are dealing with modules, these intersections are quite restricted.

Proposition 4 ([10]). *Let C be a module in G and $\mathbb{O}$ be a minimal obstruction. If $|V(\mathbb{O})| > 4$, then either $V(\mathbb{O}) \subseteq V(C)$ or $|V(\mathbb{O}) \cap V(C)| \leq 1$.*

By Proposition 4, we directly obtain the following lemma.

Lemma 4.7. *Let C be a module such that $V(C) \cap M' = \emptyset$, and let $\mathbb{O}$ be a minimal obstruction that is not covered by $\mathcal{W}$. Then, $|V(\mathbb{O}) \cap V(C)| \leq 1$.*

Proof. Since $\mathbb{O}$ is an obstruction that is not covered by $\mathcal{W}$, it holds that $|M' \cap V(\mathbb{O})| > 9$. In particular, as $V(C) \cap M' = \emptyset$, we have that $|V(\mathbb{O})| > 4$ and $V(\mathbb{O}) \setminus V(C) \neq \emptyset$. Then, as C is a module and $\mathbb{O}$ is minimal, by Proposition 4, we have that $|V(\mathbb{O}) \cap V(C)| \leq 1$. $\square$

Reducing the Size of Module Components. To ensure we have only small module components, we apply the following rule.

Reduction Rule 4.3. *Suppose that there exists $C \in \widehat{\mathcal{C}}$ such that $|V(C)| > k+1$. Then, output the instance $(G - \{v\}, k)$, where v is an arbitrarily chosen vertex of C.*

Lemma 4.8. *Reduction Rule 4.3 is safe.*

Proof. In one direction, it is clear that if (G,k) is a Yes-instance, then $(G-\{v\},k)$ is a Yes-instance as well.

In the other direction, suppose that $(G - \{v\}, k)$ is a Yes-instance. Let S be a k-solution for $G - \{v\}$. We claim that S is also a k-solution for G. Suppose, by way of contradiction, that this claim is false. Then, there exists a minimal obstruction $\mathbb{O}$ for $G - S$. As $S \cup \{v\}$ is a $(k+1)$-solution for G and $\mathcal{W}$ is $(k+2)$-necessary, we have that $S \cup \{v\}$ hits $\mathcal{W}$. Since $v \notin M$ and $\mathcal{W} \subseteq 2^M$, we derive that S hits $\mathcal{W}$. Thus, since $\mathbb{O}$ is an obstruction for $G - S$, we deduce that $\mathbb{O}$ is not covered by $\mathcal{W}$. Hence, by Lemma 4.7, $|V(\mathbb{O}) \cap V(C)| \leq 1$. Thus, $V(\mathbb{O}) \cap V(C) = \{v\}$. Then, as C is a module, for any vertex $u \in V(C)$, it holds that $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ is an obstruction. Since $|V(C)| > k+1$, we have that $V(C) \setminus (S \cup \{v\}) \neq \emptyset$. However, this implies that there exists an obstruction $\mathbb{O}'$ for $(G - \{v\}) - S$, which is a contradiction as S is a k-solution for $G - \{v\}$. $\square$

Preliminary Marking Scheme. By Lemma 4.6, for all $u, v \in M'$ such that $(u,v) \notin E(G)$ and $\{u,v\} \notin \mathcal{W}$, there exists at most one $C \in \widehat{\mathcal{C}}$, denoted by C_{uv}, such that $N_G(u) \cap N_G(v) \cap V(C) \neq \emptyset$. Accordingly, denote

$$\mathcal{C}^\star = \{C_{uv} \in \widehat{\mathcal{C}} \mid u, v \in M', (u,v) \notin E(G), \{u,v\} \notin \mathcal{W}\}.$$

Moreover, denote $A^\star = V(\mathcal{C}^\star)$. From Reduction Rule 4.3, we have the following observation.

Observation 4.9. *The size of $A^\star$ is upper bounded by $(k+1)|M'|^2$.*

Thus, in what follows, we do not need to "worry" about the modules in $\mathcal{C}^\star$ since we already know that they contain only few vertices in total. In the following, we proceed to analyze the modules in $\widehat{\mathcal{C}} \setminus \mathcal{C}^\star$. An important property of every vertex v in the modules in $\widehat{\mathcal{C}} \setminus \mathcal{C}^\star$, unlike the modules in $\mathcal{C}^\star$, is that every pair of vertices in its neighborhood in M' must be adjacent unless they form a set in $\mathcal{W}$.

Observation 4.10. *Consider a vertex $v \in V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$. For (distinct) vertices $u, w \in N_G(v) \cap M'$, at least one of $\{u,w\} \in \mathcal{W}$ or $(u,w) \in E(G)$ holds.*

Proof. For $v \in V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$, and (distinct) vertices $u, w \in N_G(v) \cap M'$, if one of $\{u,w\} \in \mathcal{W}$ or $(u,v) \in E(G)$ holds, then the claim trivially holds. Therefore, we assume that $\{u,w\} \notin \mathcal{W}$ and $(u,v) \notin E(G)$. Recall that each set in $\mathcal{W}$ is of size at least 2 (since Reduction Rule 3.1 is not applicable). From the above discussions, together with Lemma 4.6 we obtain that there is at most one connected component $C_{uw} \in \widehat{\mathcal{C}}$, such that $N_G(u) \cap N_G(w) \cap V(C_{uw}) \neq \emptyset$. Since $u, w \in N_G(v)$, it must be the case that $v \in C_{uw}$. But by our preliminary marking scheme, $C_{uw} \in \mathcal{C}^\star$. This contradicts that $v \in V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$. $\square$

Let us also consider the relation between obstructions and the modules in $\widehat{\mathcal{C}} \setminus \mathcal{C}^\star$. Roughly speaking, the following lemma already implies that we can focus on AWs of a very specific form. However, handling these obstructions requires a substantive amount of work in the rest of this section.

Lemma 4.11. *Let $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$, and $\mathbb{O}$ be a minimal obstruction that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap V(C) \neq \emptyset$. Then, $|V(\mathbb{O}) \cap V(C)| = 1$ and $\mathbb{O}$ is an AW where the vertex in $V(\mathbb{O}) \cap V(C)$ is a terminal.*

Proof. Consider $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$ and a minimal obstruction $\mathbb{O}$ that is not covered by $\mathcal{W}$, such that $V(\mathbb{O}) \cap V(C) \neq \emptyset$. First, as C is a module, from Lemma 4.7 we deduce that $|V(\mathbb{O}) \cap V(C)| = 1$. Furthermore, as $\mathbb{O}$ is not covered by $\mathcal{W}$, we have that $|V(\mathbb{O})| > 9$. This means that $\mathbb{O}$ is neither a long claw nor a whipping top. Let v be the unique vertex in $V(C) \cap V(\mathbb{O})$. If $\mathbb{O}$ is an induced cycle on at least 4 vertices, or one of the AWs where v is not one of the terminals, then $N_G(v) \cap V(\mathbb{O})$ contains a pair of non-adjacent vertices. But from Observation 4.10 together with the facts that $\mathbb{O}$ is not covered by $\mathcal{W}$ and $N_G(v) \subseteq V(C) \cup M$, for each $u, w \in N_G(v) \cap M' \cap V(\mathbb{O})$, we have $(u,v) \in E(G)$. Thus, we conclude that $\mathbb{O}$ is one of the AWs, where v is one of the terminals. $\square$

Marking Scheme to Handle Non-Shallow Terminals. For every two subsets $X, Y \subseteq M'$ such that $|X| \leq 2$ and $|Y| \leq 2$, denote $A_{X,Y} = \{v \in V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star) \mid X \subseteq N_G(v), Y \cap N_G(v) = \emptyset\}$. Now, if $|A_{X,Y}| \leq k+1$, then define $A'_{X,Y} = A_{X,Y}$, and otherwise let $A'_{X,Y}$ be an arbitrarily chosen subset of size $k+1$ of $A_{X,Y}$. Let us denote $A' = \bigcup_{X,Y} A'_{X,Y}$, where X, Y range over all subsets $X, Y \subseteq M'$ such that $|X| \leq 2$ and $|Y| \leq 2$.

Let us first observe that $|A'|$ is small (due to Reduction Rule 4.2).

Observation 4.12. *The size of A' is upper bounded by $(k+1)|M'|^4$.*

Now, let us verify that we have thus marked a set of vertices that is sufficient to "handle" non-shallow terminals. Roughly speaking, by this we mean that for any vertex v and obstruction $\mathbb{O}$ that satisfy the premise in this lemma, we can find $k+1$ "replacements" of v (so that we still have an obstruction) that belong to our marked set A'.

Lemma 4.13. *Let $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$, $v \in V(C) \setminus A'$, and $\mathbb{O}$ be a minimal obstruction that is not covered by $\mathcal{W}$ such that $v \in V(\mathbb{O})$. If $\mathbb{O}$ is not an AW where v is a non-shallow terminal, then there exists a subset $\widehat{A} \subseteq A'$ of size $k+1$ such that for each $u \in \widehat{A}$, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ contains an obstruction.*

Proof. First, by Lemma 4.11, we have that $\mathbb{O}$ is an AW such that $V(\mathbb{O}) \cap V(C) = \{v\}$ and v is a terminal of $\mathbb{O}$. Let us also note that $N_G(v) \subseteq M' \cup C$ and therefore $N_G(v) \cap V(\mathbb{O}) \subseteq M'$. Let $\mathbb{O}$ comprise of the base path $\mathsf{base}(\mathbb{O}) = (b_1, b_2, \ldots, b_z)$, non-shallow terminals t_ℓ and t_r, shallow terminal t, and centers c_1 and c_2 (as in the definition in Section 2). Here, if $\mathbb{O}$ is a †-AW, then we let $c = c_1 = c_2$. Suppose that v is not the shallow terminal of $\mathbb{O}$. Then, we have that v is either t_ℓ or t_r. Without loss of generality, suppose that $v = t_\ell$. Let us consider two cases, depending on whether $\mathbb{O}$ is a †-AW or a ‡-AW.

- Suppose that $\mathbb{O}$ is a †-AW. Notice that $b_1 \in M'$ as $(b_1, v) \in E(G)$, $V(\mathbb{O}) \cap V(C) = \{v\}$, and $N_G(v) \subseteq M' \cup C$. From Lemma 4.11 any vertex in $V(\mathbb{O}) \cap V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$ must be one of the terminals. Thus, we have $V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star) \cap (\{b_1, b_2, \ldots, b_z\} \cup \{c\}) = \emptyset$. We also recall that for each $u \in V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$, we have $N_G(u) \subseteq M' \cup V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$. In particular, if b_2 (or c) is not in M', no vertex in $V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$ can be adjacent to b_2 (or c). The above discussions together with the construction of A' implies the following: there exists a subset $Q \subseteq A'$ of $k+1$ vertices such that for each $u \in Q$, u is adjacent to b_1, and u is not adjacent to b_2 and c. Indeed, these are the vertices in the set $A'_{\{b_1\},\{b_2,c\}\cap M'}$ (the size of this set is $k+1$ since otherwise v should have belonged to it, but $v \notin A'$). Furthermore, b_1 is not adjacent to any vertex on $\mathbb{O}$ besides v, c and b_2. Therefore, for all $u \in Q$, using Observation 4.10 for obstructions not covered by $\mathcal{W}$, we have that u is not adjacent to any vertex on $V(\mathbb{O}) \cap M'$ besides b_1. Furthermore, for all $u \in Q$, since $N_G(u) \subseteq V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star) \cup M'$, we have that u is not adjacent to any vertex on $V(\mathbb{O}) \cap V(\mathcal{C}^\star)$. Lastly, because $V(\mathbb{O}) \cap V(C) = \{v\}$, for all $u \in Q$, we have that u is not adjacent to any vertex on $V(\mathbb{O}) \cap V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$ besides possibly v. Hence, for any vertex $u \in Q$, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ is also a †-AW.

- Suppose that $\mathbb{O}$ is a ‡-AW. Notice that $b_1, c_1 \in M'$ as $(b_1, v), (c_1, v) \in E(G)$, $V(\mathbb{O}) \cap V(C) = \{v\}$, and $N_G(v) \subseteq M' \cup C$. From Lemma 4.11 any vertex in $V(\mathbb{O}) \cap V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$ must be one of the terminals. Thus, we have $V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star) \cap (\{b_1, b_2, \ldots, b_z\} \cup \{c\}) = \emptyset$. We also recall that for each $u \in V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$, we have $N_G(u) \subseteq M' \cup V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star)$. The above discussions together with the construction of A' implies the following: there exists a subset $Q \subseteq A'$ of $k+1$ vertices $u \in A'$ such that u is adjacent to both c_1 and b_1, and u is adjacent to neither c_2 nor b_2. Indeed, these are the vertices in the set $A'_{\{b_1,c_1\},\{b_2,c_2\}\cap M'}$ (as in the previous case, the size of this set is $k+1$ since otherwise v should have belonged to it, but $v \notin A'$). Notice that b_1 is not adjacent to any vertex on $\mathbb{O}$ besides v, c_1, c_2 and b_2. For all $u \in Q$, using Observation 4.10 for obstructions not covered by $\mathcal{W}$ and the facts that $N_G(u) \subseteq V(\widehat{\mathcal{C}} \setminus \mathcal{C}^\star) \cup M'$ and $V(\mathbb{O}) \cap V(C) = \{v\}$ (using the exact same rationale as in the previous case), we have that u is not adjacent to any vertex on $\mathbb{O} - \{v\}$ besides c_1 and b_1. Hence, for any vertex $u \in Q$, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ is also a ‡-AW.

In both cases, we derived the desired claim, and thus the proof is complete. □

Marking Scheme to Handle Shallow Terminals. For this part in our proof, we require the following notation: we say that a path P is *covered by* $\mathcal{W}$ if there is a set $W \in \mathcal{W}$ such that $W \subseteq V(P)$. Intuitively, we think of P as part of the base of an obstruction, hence the notation above is a natural extension of covering to this context.

Before we present our marking scheme, let us explicitly state the following observation, which follows from Observation 4.10 in the same manner as Lemma 4.11.

Observation 4.14. *Let P be an induced path in $G[V(G) \setminus V(C)]$ for some $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$ such that P is not covered by $\mathcal{W}$. For all $v \in V(C)$, $|N_G(v) \cap V(P)| \leq 2$, and if $|N_G(v) \cap V(P)| = 2$, then the two vertices in $N_G(v) \cap V(P)$ are adjacent on P.*

Proof. Consider $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$, $v \in V(C)$, and an induced path P in $G[V(G) \setminus V(C)]$ which is not covered by $\mathcal{W}$. If $|N_G(v) \cap V(P)| \leq 1$, then the claim trivially follows. Otherwise, we assume that $|N_G(v) \cap V(P)| \geq 2$. Consider (distinct) vertices $u, w \in N_G(v) \cap V(P)$. From Observation 4.10, we have that $(u, w) \in E(G)$. Here, we relied on the fact that P is not covered by $\mathcal{W}$. Since P is an induced path, u and w must be adjacent vetices in P. From the above we can conclude that v cannot have three neighbors in P as P is an induced path in G. Moreover, if v has two neighbors in P then they must be adjacent vertices. ☐

Denote $N = M' \cup A^\star \cup A'$. (Recall that $A^\star = V(\mathcal{C}^\star)$ and that A' is the set of vertices marked when we dealt with non-shallow terminals.) For all (not necessarily distinct) vertices $c_1, c_2 \in M'$, denote $A_{\{c_1,c_2\}} = \{v \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A') \mid \{c_1, c_2\} \subseteq N_G(v)\}$. Intuitively, $A_{\{c_1,c_2\}}$ is the set of vertices among the unmarked vertices in $\widehat{\mathcal{C}}$ that are neighbors of both c_1 and c_2 and hence can play the role of shallow terminals in obstructions having c_1 and c_2 as centers. Moreover, let us arbitrarily order N and $E(G[N])$ as follows: $N = \{v_1, v_2, \ldots, v_{|N|}\}$ and $E(G[N]) = \{e_1, e_2, \ldots, e_{|E(G[N])|}\}$. Thus, when we define vectors having $|N|$ or $|E(G[N])|$ entries below, we can work with a natural correspondence between the index of an entry in the vector and an element of N or $E(G[N])$, respectively.

In what follows, we begin the part in our analysis that is based on linear algebra. To this end, we first need to encode our problem in this language, which entails the introduction of appropriate notations. Afterwards, we will present a marking scheme based on these notations. The analysis of this scheme is done is a sequence of several lemmata, after which we will be ready to conclude the proof of Lemma 4.3.

First, with every vertex $u \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A')$, we associate two binary vectors that capture incidence relations between u and the elements (vertices and edges) in $G[N]$:

- **Vertex incidence relations.** $\mathsf{vinc}(u) = (b_1, b_2, \ldots, b_{|N|})$, where for all $i \in [|N|]$, $b_i = 1$ if and only if $v_i \in N_G(u)$;
- **Edge incidence relations.** $\mathsf{einc}(u) = (b_1, b_2, \ldots, b_{|E(G[N])|})$, where for all $i \in [|E(G[N])|]$, $b_i = 1$ if and only if u is adjacent to both endpoints of e_i.

Complete incidence relations. In addition, we define $\mathsf{inc}(u)$ as the vector that is the concatenation of $\mathsf{vinc}(u)$ and $\mathsf{einc}(u)$, to which we add 1 at the end. Formally, $\mathsf{inc}(u)$ is a binary vector with $|N| + |E(G[N])| + 1$ entries, where for all $i \in [|N|]$, the i^{th} entry of $\mathsf{inc}(u)$ equals the i^{th} entry of $\mathsf{vinc}(u)$, for all $i \in [|E(G[N])| + |N|] \setminus [|N|]$, the i^{th} entry of $\mathsf{inc}(u)$ equals the $(i-|N|)^{th}$ entry of $\mathsf{einc}(u)$, and the last entry of $\mathsf{inc}(u)$ is 1. These incidence vectors are associated with the vector space $\mathbb{F}_2^q$ for $q = |N| + |E(G[N])| + 1$, and all calculations related to these vectors are performed accordingly. This completes the description of the notations required to present our marking scheme.

For all (not necessarily distinct) vertices $c_1, c_2 \in M'$, we have the following subprocedure of our marking scheme. First, we define $\mathbf{V}_{\{c_1,c_2\}}$ to be the *multiset* $\{\mathsf{inc}(u) \mid u \in A_{\{c_1,c_2\}}\}$. More precisely, the number of occurrences of a vector in $\mathbf{V}_{\{c_1,c_2\}}$ equals the number of vertices $u \in A_{\{c_1,c_2\}}$ such that $\mathsf{inc}(u)$ equals that vector. Now, we proceed as follows.

1. Initialize $\widehat{\mathbf{V}}^0_{\{c_1,c_2\}} = \emptyset$.
2. For $i = 1, 2, \ldots, k+1$: compute some basis $\mathbf{B}^i_{\{c_1,c_2\}}$ for the vector subspace $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$ (with respect to $\mathbb{F}_2^q$),[2] and denote $\widehat{\mathbf{V}}^i_{\{c_1,c_2\}} = \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}} \cup \mathbf{B}^i_{\{c_1,c_2\}}$.

[2]Here, note that the subtraction concerns multisets. In particular, if an element occurs x times in a multiset X, and y times in a multiset $Y \subseteq X$, then it occurs $x - y$ times in $X \setminus Y$.

3. For every occurrence of a vector $\mathbf{v} \in \widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}$, arbitrarily choose a unique vertex $u \in A_{\{c_1,c_2\}}$ such that $\mathsf{inc}(u) = \mathbf{v}$ and denote it by $u_{\mathbf{v}}$ (the existence of sufficiently many such distinct vertices directly follows from the definition of $\mathbf{V}_{\{c_1,c_2\}}$).

4. Denote $\widehat{A}_{\{c_1,c_2\}} = \{u_{\mathbf{v}} : \mathbf{v} \in \widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}\}$, and note that $\widehat{A}_{\{c_1,c_2\}}$ is a set (rather than a multiset).

Finally, having performed all subprocedures, we denote $\widehat{A} = \bigcup_{c_1,c_2 \in M'} \widehat{A}_{\{c_1,c_2\}}$. Here, union refers to sets, that is, every vertex occurs in $\widehat{A}$ once even if it belongs to more than one set of the form $\widehat{A}_{\{c_1,c_2\}}$. This completes the description of our marking scheme.

We proceed to analyze our marking scheme. Let us first observe that we have not marked "many" vertices, that is, we upper bound $|\widehat{A}|$. (Here, the bound $|N| \leq 2|M'|^4$ follows from Observations 4.9 and 4.12, and since $N = M' \cup A^\star \cup A'$.)

Lemma 4.15. *The size of $\widehat{A}$ is upper bounded by $(k+1)|M'|^2|N|^2 \leq 2(k+1)^2|M'|^6$.*

Proof. To show that $|\widehat{A}| \leq (k+1)|M'|^2|N|^2$, it is sufficient to show that for all $c_1, c_2 \in M'$, $|\widehat{A}_{\{c_1,c_2\}}| \leq (k+1)|N|^2$. To this end, consider some $c_1, c_2 \in M'$. Now, observe that the number of entries of the vectors in $\mathbf{V}_{\{c_1,c_2\}}$ is $q = |N| + |E(G[N])| + 1 \leq |N| + \frac{|N|(|N|-1)}{2} + 1 \leq |N|^2$ (assuming $|N| > 1$, as otherwise, we can obtain a trivial kernel). Hence, every basis of $\mathbf{V}_{\{c_1,c_2\}}$ (or of a subset of $\mathbf{V}_{\{c_1,c_2\}}$) is of size at most $|N|^2$. As $\widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}$ is a multiset that is the union of $(k+1)$ bases of $\mathbf{V}_{\{c_1,c_2\}}$ (or of subsets of $\mathbf{V}_{\{c_1,c_2\}}$), we have that $|\widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}| \leq (k+1)|N|^2$. Since $|\widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}| = |\widehat{A}_{\{c_1,c_2\}}|$, the proof is complete. $\square$

Now, let us verify that we have a set of vertices that is sufficient to "handle" shallow terminals. This will be done in a sequence of two lemmata and a corollary. For this purpose, we need the following notation where we alter incidence vectors by nullifying some of their entries.

- **Nullifying Subsets of Vertices and Edges.** Given a pair (X, Y), where $X \subseteq N$ and $Y \subseteq E(G[N])$, and a vertex $u \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A')$, we define $\mathsf{inc}^{X,Y}(u)$ to be the vector $\mathsf{inc}(u)$ where all entries associated with vertices and edges that do not belong to $X \cup Y$ are changed to 0. Formally, $\mathsf{inc}^{X,Y}(u)$ is a binary vector with $|N| + |E(G[N])| + 1$ entries, where for all $i \in [|N|]$, the i^{th} entry of $\mathsf{inc}(u)$ equals the i^{th} entry of $\mathsf{vinc}(u)$ if $v_i \in X$ and to 0 otherwise, for all $i \in [|E(G[N])| + |N|] \setminus [|N|]$, the i^{th} entry of $\mathsf{inc}^{X,Y}(u)$ equals the $(i - |N|)^{th}$ entry of $\mathsf{einc}(u)$ if $e_{i-|N|} \in Y$ and to 0 otherwise, and the last entry of $\mathsf{inc}^{X,Y}(u)$ is 1.

- **Nullifying an Induced Path.** Furthermore, for an induced path P in $G - (V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A'))$ and a vertex $u \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A')$, we denote $\mathsf{inc}^P(u) = \mathsf{inc}^{X,Y}(u)$ where $X = V(P) \cap N$ and $Y = E(P) \cap E(G[N])$.

Moreover, recall that given a vector $\mathbf{v}$ and an entry index i, $\mathbf{v}[i]$ denotes the i^{th} entry of $\mathbf{v}$.

Lemma 4.16. *Let P be an induced path in $G[V(G) \setminus V(C)]$ for some $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$ such that P is not covered by $\mathcal{W}$. For all $u \in V(C)$, $\sum_{i=1}^{q} \mathsf{inc}^P(u)[i] = 1 \mod 2$ if and only if $N_G(u) \cap V(P) = \emptyset$.*

Proof. Consider some vertex $u \in V(C)$. For the reverse direction of the proof, suppose that $N_G(u) \cap V(P) = \emptyset$. Then, all of the entries of $\mathsf{inc}^P(u)$ equal 0, except for the last entry which equals 1. Thus, $\sum_{i=1}^{q} \mathsf{inc}^P(u)[i] = 1 \mod 2$.

For the forward direction of the proof, suppose that $N_G(u) \cap V(P) \neq \emptyset$. Then, by Observation 4.14, $|N_G(u) \cap V(P)|$ is either 1 or 2, and if it is 2, then the two vertices in $N_G(u) \cap V(P)$ are adjacent on P. Furthermore, observe that as $V(P) \cap V(C) = \emptyset$ and $N_G(u) \subseteq V(C) \cup M'$, we have that $N_G(u) \cap V(P) \subseteq M'$. Thus, in case $|N_G(u) \cap V(P)| = 1$, it follows that there exists exactly one entry in $\mathsf{inc}^P(u)$ that equals 1 apart from the last entry, which is the entry corresponding to the vertex in $N_G(u) \cap V(P)$. Moreover, in case $|N_G(u) \cap V(P)| = 2$, it follows that there exist exactly three entries in $\mathsf{inc}^P(u)$ that equal 1 apart from the last entry, which are the two entries corresponding to the two vertices in $N_G(u) \cap V(P)$ and the entry corresponding to the edge between these two vertices. In both cases, we derive that $\sum_{i=1}^{q} \mathsf{inc}^P(u)[i] = 0 \mod 2$ as desired. $\square$

The reason why we need Lemma 4.16 is that we make use of it in the proof of the following lemma. Informally, this lemma exhibits the existence of $k+1$ "replacements" for each unmarked shallow terminal.

Lemma 4.17. *Let $w \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A})$, and $\mathbb{O}$ be an AW that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap (V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$ and w is the shallow terminal of $\mathbb{O}$. Let $\{c_1, c_2\}$ be the set of centers of $\mathbb{O}$ (with $c_1 = c_2$ if $\mathbb{O}$ is a $\dagger$-AW). Then, for all $i \in [k+1]$, there exists $\mathbf{v} \in \mathbf{B}^i_{\{c_1,c_2\}}$ such that $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u_\mathbf{v}\}]$ is an obstruction.*

Proof. Consider some $i \in [k+1]$. Let C be the connected component in $\widehat{\mathcal{C}}$ containing w. Notice that $c_1, c_2 \in M'$ as $(c_1, w), (c_2, w) \in E(G)$, $V(\mathbb{O}) \cap (V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$, and $N_G(w) \subseteq M' \cup C$. Let us first argue that there exists an occurrence of $\mathsf{inc}(w)$ in $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$. To this end, note that as w is the shallow terminal of $\mathbb{O}$, it is adjacent to c_1 and c_2, and therefore $w \in A_{\{c_1,c_2\}}$. Moreover, because $w \notin \widehat{A}$, there exists an occurrence of $\mathsf{inc}(w)$ that does not belong to $\mathbf{V}^{k+1}_{\{c_1,c_2\}}$, which implies that there exists an occurrence of $\mathsf{inc}(w)$ in $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$.

As we have shown that $\mathsf{inc}(w)$ in $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$, the fact that $\mathbf{B}^i_{\{c_1,c_2\}}$ is a basis for $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$ implies that there exist vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$ for some $t \in \mathbb{N}$ (in particular, $t \geq 1$) and nonzero coefficients $\lambda_1, \lambda_2, \ldots, \lambda_t$ such that $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_t \mathbf{v}_t = \mathsf{inc}(w)$ over $\mathbb{F}_2^q$. As the coefficient are from field $\mathbb{F}_2$, they are all necessarily 1. Thus, we have that

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_t = \mathsf{inc}(w) \text{ over } \mathbb{F}_2^q.$$

Denote $u_i = u_{\mathbf{v}_i}$ for all $i \in [t]$. Then, $\mathsf{inc}(u_1) + \mathsf{inc}(u_2) + \cdots + \mathsf{inc}(u_t) = \mathsf{inc}(w)$ over $\mathbb{F}_2^q$. In particular, $\mathsf{inc}^P(u_1) + \mathsf{inc}^P(u_2) + \cdots + \mathsf{inc}^P(u_t) = \mathsf{inc}^P(w)$ over $\mathbb{F}_2^q$, where P is the extended base of $\mathbb{O}$. This implies that $\sum_{i=1}^{t} \sum_{j=1}^{q} \mathsf{inc}^P(u_i)[j] = \sum_{j=1}^{q} \mathsf{inc}^P(w)[j] \mod 2$. (Note that since $V(\mathbb{O}) \cap (V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$, the extended base is completely contained in $G[V(G) \setminus (V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A}))]$, and furthermore P is not covered by $\mathcal{W}$ by the premise of the lemma.) By Lemma 4.16 and since $N_G(w) \cap V(P) = \emptyset$ (because w is the shallow terminal of $\mathbb{O}$), we have that $\sum_{j=1}^{q} \mathsf{inc}^P(w)[j] = 1 \mod 2$. Thus, $\sum_{i=1}^{t} \sum_{j=1}^{q} \mathsf{inc}^P(u_i)[j] = 1 \mod 2$. This implies that there exists $i \in [t]$ such that $\sum_{j=1}^{q} \mathsf{inc}^P(u_i)[j] = 1 \mod 2$. However, by Lemma 4.16, this means that $N_G(u_i) \cap V(P) = \emptyset$. Moreover, we have that $u_i \in A_{\{c_1,c_2\}}$ because u_i is associated with the vector $\mathbf{v}_i$ which belongs to $\mathbf{B}^i_{\{c_1,c_2\}}$. Hence, $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u_i\}]$ is an AW. This completes the proof. $\square$

Due to the definition of $\widehat{A}$, as a direct corollary to Lemma 4.17 we have the following result.

Corollary 4.18. *Let $w \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A})$, and $\mathbb{O}$ be an AW that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap (V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$ and w is the shallow terminal of $\mathbb{O}$. Then, there exists a set $\tilde{A} \subseteq \widehat{A}$ of size $k+1$ such that for each $u \in \tilde{A}$, $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u\}]$ is an obstruction.*

We are now ready to conclude the proof of Lemma 4.3 and thereby this section.

Proof of Lemma 4.3. Towards the proof, first note that if the condition of Reduction Rule 4.3 applies, then we are clearly done—indeed, in this case we output an instance (G', k) equivalent to

(G,k) where G' is a strict subgraph of G. Thus, we next suppose that this rule has been applied exhaustively. Then, our output is the set $B = A^\star \cup A' \cup \widehat{A}$. By Observations 4.9 and 4.12, and by Lemma 4.15, we have that $|B| \leq |A^\star| + |A'| + |\widehat{A}| \leq (k+1)|M'|^2 + (k+1)|M'|^4 + 2(k+1)^2|M'|^6 \leq 4(k+1)^2|M'|^6$ as desired.

Let $S \subseteq V(G)$ be some arbitrary set of size at most k. We claim that the following property holds: If there exists an obstruction $\mathbb{O}$ for G that is not covered by $\mathcal{W}$ and such that $V(\mathbb{O}) \cap S = \emptyset$, then there exists an obstruction $\mathbb{O}'$ for G such that $V(\mathbb{O}') \cap S = \emptyset$ and $V(\mathbb{O}') \cap (V(\widehat{\mathcal{C}}) \setminus B) = \emptyset$. Clearly, if there does not exist any obstruction $\mathbb{O}$ for G that is not covered by $\mathcal{W}$ and such that $V(\mathbb{O}) \cap S = \emptyset$, then our proof is complete. Hence, we next suppose that such an obstruction exists, and we let $\mathbb{O}'$ be such a minimal obstruction that minimizes $|V(\mathbb{O}') \cap (V(\widehat{\mathcal{C}}) \setminus B)|$. We claim that for this obstruction $\mathbb{O}'$, it holds that $V(\mathbb{O}') \cap (V(\widehat{\mathcal{C}}) \setminus B) = \emptyset$, which would complete the proof. Suppose, by way of contradiction, that this claim is false. Then, as $V(\mathcal{C}^\star) \subseteq B$, there exists $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$ and $v \in V(C)$ such that $v \in V(\mathbb{O}')$. By Lemma 4.11, $|V(\mathbb{O}) \cap V(C)| = 1$ and $\mathbb{O}'$ is an AW where v is a terminal.

Let us first suppose that v is not the shallow terminal of $\mathbb{O}'$. Then, by Lemma 4.13, there exist $(k+1)$ vertices $u \in A'$ such that $G[(V(\mathbb{O}') \setminus \{v\}) \cup \{u\}]$ is an obstruction. However, as $|S| \leq k$, this means that there exists $u \in A' \setminus S$ such that $G[(V(\mathbb{O}') \setminus \{v\}) \cup \{u\}]$ is an obstruction. As $A' \subseteq B$ and $G[(V(\mathbb{O}') \setminus \{v\}) \cup \{u\}]$ has fewer vertices from $V(\widehat{\mathcal{C}}) \setminus B$ than $\mathbb{O}'$, we have reached a contradiction to the choice of $\mathbb{O}$.

As the choice of v was arbitrary, we derive that $V(\mathbb{O}') \cap (V(\widehat{\mathcal{C}}) \setminus B)$ contains exactly one vertex, which we denote by w, that is the shallow terminal of $\mathbb{O}'$. In this case, by Corollary 4.18, there exist $(k+1)$ vertices $u \in \widehat{A}$ such that $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u\}]$ is an obstruction. However, as $|S| \leq k$, this means that there exists $u \in \widehat{A} \setminus S$ such that $G[(V(\mathbb{O}') \setminus \{w\}) \cup \{u\}]$ is an obstruction. As $\widehat{A} \subseteq B$ and $G[(V(\mathbb{O}') \setminus \{w\}) \cup \{u\}]$ has no vertices from $V(\widehat{\mathcal{C}}) \setminus B$, we have again reached a contradiction to the choice of $\mathbb{O}$. This completes the proof. $\square$

4.1 Bounded Intersection Two Families Lemma

At the heart of our marking scheme to handle shallow terminals is in fact the special case of Lemma 1.1 where $c = 2$. Indeed, viewing this case in a more abstract manner, let us give a rough description of the relation between it and the statement of Lemma 1.1. For all $c_1, c_2 \in M'$, we have sets $A_1, A_2, \ldots, A_t$ and $B_1, B_2, \ldots, B_t$, that are defined as follows. First, the universe is the set of all vertices and pairs of vertices in N. Second, let W denote a set of vertices $w \in V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A')$ such that **(i)** w is adjacent to c_1 and c_2, and **(ii)** w has at least one induced path in $G[N]$, say P_w, which contains no vertex adjacent to w, and so that the two following properties hold:

- For all distinct $w, w' \in W$, w is adjacent to at least one vertex on $P_{w'}$.
- For every induced path P in $G[N]$ that has no vertex adjacent to some vertex in $V(\widehat{\mathcal{C}}) \setminus (A^\star \cup A')$, there also exists a vertex in W that is not adjacent to any vertex on P.

These properties mean, in a sense, that W is a minimal set that "covers" all induced paths in $G[N]$ that can potentially create AWs together with c_1 and c_2 as centers. Then, $t = |W|$, and denote $W = \{w_1, w_2 \ldots, w_t\}$. For every vertex $w_i \in W$, we create the new set A_i, which contains all the neighbors of w_i in N, and the new set B_i, which is equal to $V(P_{w_i})$. Clearly, for all $i \in [t]$, $A_i \cap B_i = \emptyset$, and due to Observation 4.14, for all distinct $i, j \in [t]$, $|A_i \cap B_i| \in \{1, 2\}$.

Let us now turn to the proof of Lemma 1.1. For convenience, let us restate it.

Lemma 1.1 (Bounded Intersection Two Families Lemma)**.** *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be families over a universe U such that (i) for every $i \leq m$, $A_i \cap B_i = \emptyset$, and (ii) for every $j \neq i$, $|A_i \cap B_j| \in \{1, \ldots, c\}$. Then $m \leq \sum_{t=0}^{c} \binom{|U|}{t}$.*

Proof. Let $|U| = n$ and let $d = \sum_{t=0}^{c} \binom{n}{t}$. Let D be the set of all subsets of U of size at most c (including the empty set). Note that we have $|D| = d$. Fix a bijection between D and $\{1, 2, \ldots, d\}$. We construct an incidence vector $\mathbf{v}_i$ for each set A_i, where $\mathbf{v}_i$ is indexed by the subsets of U of size up to c. More precisely, we have a vector $\mathbf{v}_i \in \{0,1\}^d$, where $\mathbf{v}_i[X] = 1$ if and only if $X \subseteq A_i$. Let us note that $\mathbf{v}_i[\emptyset] = 1$ for all $1 \leq i \leq m$. We consider these vectors as elements of the vector space $\mathbb{F}_2^d$. Similarly, we construct vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ for each set $B_1, B_2, \ldots, B_m$. We first claim that for every $i \in [m]$, we have $\mathbf{v}_i \cdot \mathbf{u}_i = 1$. This follows from the fact that $A_i \cap B_i = \emptyset$. We next claim that, for each $i, j \in [m]$, where $i \neq j$, we have $\mathbf{v}_i \cdot \mathbf{u}_j = 0$. This follows from the following observation. Let $C_{ij} = A_i \cap B_j$. Then, as $|C_{ij}| \in [c]$, we have that $2^{C_{ij}} \subseteq D$, where $2^{C_{ij}}$ denotes the collection of all subsets of C_{ij}. Now, observe that $\mathbf{v}_i[X]\mathbf{u}_j[X] = 1$ if and only if $X \subseteq C_{ij}$. As $|2^{C_{ij}}|$ is an even number (greater than or equal to 2), it follows that $\mathbf{v}_i \cdot \mathbf{u}_j = 0$ over the field $\mathbb{F}_2$.

Now suppose that $m > d$. Then the collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is not linearly independent in $\mathbb{F}_2^d$. Hence, there is a vector, say $\mathbf{v}_m$, such that $\mathbf{v}_m = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_{m-1} \mathbf{v}_{m-1}$, where $\alpha_j \in \mathbb{F}_2$ for each $j \in [m-1]$. We claim that there is a vector $\mathbf{v}_i$ such that $\mathbf{v}_i \cdot \mathbf{u}_m = 1$ for some $i \in [m-1]$. This follows from the following equation.

$$\mathbf{v}_m \cdot \mathbf{u}_m = (\sum_{j=1}^{m-1} \alpha_j \mathbf{v}_j) \cdot \mathbf{u}_m$$
$$\implies 1 = \sum_{j=1}^{m-1} \alpha_j (\mathbf{v}_j \cdot \mathbf{u}_m)$$

However, this is a contradiction. Hence, $m \leq d$. This concludes the proof of this lemma. $\square$

5 Bounding the Maximum Size of a Clique of Non-module Components

Let $\eta = 2^{10} \cdot 4(k+5)\binom{|M|}{10}$. Recall that $\mathcal{C}$ is the set of connected components of $G - M$, $\mathcal{D}$ is the set of connected components in $\mathcal{C}$ that are modules, and $\overline{\mathcal{D}} = \mathcal{C} \setminus \mathcal{D}$. Let $(\mathbb{P}, \beta)$ be a clique path of $G[V(\overline{\mathcal{D}})]$, $V(\mathbb{P}) = \{x_1, x_2, \ldots, x_t\}$, and for each $i \in [t]$ we let $B_i = \beta(x_i)$. Furthermore, let $\beta(\mathbb{P}) = \cup_{i=1}^{t} \beta(x_i)$. Let B_i be a bag such that $|B_i| > \eta$. Towards bounding the size of B_i, we mark some of the vertices in B_i, and delete all the unmarked vertices in B_i from G. In fact, in a step we *only delete* one unmarked vertex, and then repeat the whole kernelization algorithm on the reduced instance. In the following, we describe the precise marking procedure.

Marking Scheme. To define our marking scheme, we first introduce some notations. We define two functions namely, $\mathsf{id}_\ell^i, \mathsf{id}_r^i : B_i \to [t]$. Intuitively, these functions denote how far or close a vertex appears in the bags that are to the left and right of B_i, respectively. For a vertex $v \in B_i$, $\mathsf{id}_\ell^i(v)$ is the smallest integer $x \in [t]$ such that $v \in B_x$, and $\mathsf{id}_r^i(v)$ is the largest integer $y \in [t]$ such that $v \in B_y$. Note that for each $v \in B_i$, we have $\mathsf{id}_\ell^i(v) \leq i \leq \mathsf{id}_r^i(v)$. A frame $\mathbb{F} = (X, Y)$ in G is a pair of vertex subsets, such that $X \subseteq M$ of size at most 10 and $Y \subseteq X$. A vertex $v \in V(G)$ is said to *fit* a frame $\mathbb{F} = (X, Y)$ if $N_G(v) \cap X = Y$. We now move to the construction of the set $H_i \subseteq B_i$, of marked vertices. For each frame $\mathbb{F}$ in G, we create four sets $L_{\mathsf{far}}^{\mathbb{F},i}, L_{\mathsf{cls}}^{\mathbb{F},i}, R_{\mathsf{far}}^{\mathbb{F},i}, R_{\mathsf{cls}}^{\mathbb{F},i} \subseteq B_i$ of marked vertices each of size as most $k+5$ (and add these vertices to H_i) as follows.

- We create the set $L_{\mathsf{far}}^{\mathbb{F},i}$ as follows. Let W be the set of unmarked vertices in B_i, that fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $L^{\mathbb{F},i}_{\mathsf{far}}$. Else, let $W_{\mathsf{low}} \subseteq W$ be the set of $k+5$ vertices with lowest id^i_ℓ values among the vertices in W. Add W_{low} to $L^{\mathbb{F},i}_{\mathsf{far}}$.

- We create the set $L^{\mathbb{F},i}_{\mathsf{cls}}$ as follows. Let W be the set of unmarked vertices in B_i, that fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $L^{\mathbb{F},i}_{\mathsf{cls}}$. Else, let $W_{\mathsf{high}} \subseteq W$ be the set of $k+5$ vertices with highest id^i_ℓ values among the vertices in W. Add W_{high} to $L^{\mathbb{F},i}_{\mathsf{cls}}$.
- We create the set $R^{\mathbb{F},i}_{\mathsf{far}}$ as follows. Let W be the set of unmarked vertices in B_i, that fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $R^{\mathbb{F},i}_{\mathsf{far}}$. Else, let $W_{\mathsf{high}} \subseteq W$ be the set of $k+5$ vertices with highest id^i_r values among the vertices in W. Add W_{high} to $R^{\mathbb{F},i}_{\mathsf{far}}$.
- We create the set $R^{\mathbb{F},i}_{\mathsf{cls}}$ as follows. Let W be the set of unmarked vertices in B_i, that fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $R^{\mathbb{F},i}_{\mathsf{cls}}$. Else, let $W_{\mathsf{low}} \subseteq W$ be the set of $k+5$ vertices with lowest id^i_r values among the vertices in W. Add W_{low} to $R^{\mathbb{F},i}_{\mathsf{cls}}$.

Notice that $|H_i| \leq 2^{10} \cdot 4(k+5)\binom{|M|}{10} = \eta$. Before proceeding further, we observe (Observation 5.1 and 5.2) certain useful properties regarding a frame $\mathbb{F}$ to which $v \in B_i \setminus H_i$ fits and the vertices in $L^{\mathbb{F},i}_{\mathsf{far}}, R^{\mathbb{F},i}_{\mathsf{far}}, L^{\mathbb{F},i}_{\mathsf{cls}}$, and $R^{\mathbb{F},i}_{\mathsf{cls}}$.

Observation 5.1. *For a frame $\mathbb{F} = (X, Y)$ to which v fits and a vertex $w \in N_G(v)$ the following holds.*

- *If $w \in Y$, then $L^{\mathbb{F},i}_{\mathsf{far}} \cup R^{\mathbb{F},i}_{\mathsf{far}} \subseteq N_G(w)$.*
- *If $w \in V(G) \setminus M$, then at least one of $L^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$ or $R^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$ holds.*

Proof. In the first case, it follows from the definition that $L^{\mathbb{F},i}_{\mathsf{far}} \cup R^{\mathbb{F},i}_{\mathsf{far}} \subseteq N_G(w)$. Now we prove the second part of the observation. First, consider the case when both v and w belong to B_i. In this case second claim holds, because B_i is a clique, $L^{\mathbb{F},i}_{\mathsf{far}} \subseteq B_i$ and $R^{\mathbb{F},i}_{\mathsf{far}} \subseteq B_i$. So let us assume that $w \notin B_i$. However, $w \in N_G(v)$ and hence both v and w lie in the same bag, say B_j, on the clique path $\mathbb{P}$. Since the bags in which w is present occur consecutively on $\mathbb{P}$, we have that all these bags either appear left of B_i or right of B_i. Let us consider the case when all the bags containing w appear left of B_i. The other case when all the bags containing w appear right of B_i is symmetric. We will show that $L^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$. Towards this we will show that for every $x \in^{\mathbb{F},i}_{\mathsf{far}} \setminus\{w\}$, there exists a bag that contains both x and w. For a vertex z, let s_z denote the leftmost bag on $\mathbb{P}$ in which z appears and e_z denote the rightmost bag on $\mathbb{P}$ in which z appears. Recall that v is an unmarked vertex in B_i and thus, $s_x \leq s_v \leq i \leq e_x$. Furthermore, we know that $s_x \leq j < i$. This implies that x also belongs to B_j. Hence, we have shown that $L^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$. This concludes the proof. □

Observation 5.2. *For a frame $\mathbb{F} = (X, Y)$ to which v fits and a vertex $w \notin N_G(v)$ the following holds.*

- *If $w \in X \setminus Y$, then $(L^{\mathbb{F},i}_{\mathsf{cls}} \cup R^{\mathbb{F},i}_{\mathsf{cls}}) \cap N_G(w) = \emptyset$.*
- *If $w \in V(G) \setminus M$, then at least one of $L^{\mathbb{F},i}_{\mathsf{cls}} \cap N_G(w) = \emptyset$ or $R^{\mathbb{F},i}_{\mathsf{cls}} \cap N_G(w) = \emptyset$ holds.*

Proof. In the first case, it follows from the definition that $(L^{\mathbb{F},i}_{\mathsf{cls}} \cup R^{\mathbb{F},i}_{\mathsf{cls}}) \cap N_G(w) = \emptyset$. In the second case, if $w \notin V(\overline{\mathcal{D}})$ then the claim trivially holds. Otherwise, v and w lie in the clique path $\mathbb{P}$. Since $w \notin N_G(v)$, there is no bag which contains both v and w, and $v \in B_i$. Either w appears only in the bags (strictly) to the left of B_i, in which case v being an unmarked vertex implies that $L^{\mathbb{F},i}_{\mathsf{cls}} \cap N_G(w) = \emptyset$. On the other hand, if w appears only in the bags (strictly) to the right of B_i, we have $R^{\mathbb{F},i}_{\mathsf{cls}} \cap N_G(w) = \emptyset$. □

Next, we give a reduction rule that deletes unmarked vertices from B_i in G.

Reduction Rule 5.1. *Let v be a vertex in $B_i \setminus H_i$. Delete v from G i.e., the resulting instance is $(G - \{v\}, k)$.*

Lemma 5.3. *Reduction Rule 5.1 is safe.*

Before moving to the proof of Lemma 5.3, we note that using it we immediately obtain the following lemma.

Lemma 5.4. *If Reduction Rule 5.1 is not applicable, then for each $j \in [t]$, we have $|B_j| \leq \eta$.*

Proof. Follows from the safeness of Reduction Rule 5.1 (Lemma 5.3) and the fact that $|H_j| \leq \eta$, for each $j \in [t]$. $\square$

In the remainder of this section we focus on the proof of Lemma 5.3. Let v be a vertex in $B_i \setminus H_i$ and $G' = G - \{v\}$. We will show that (G, k) is a Yes instance of IVD if and only if (G', k) is a Yes instance of IVD. In the forward direction, let S be a solution to IVD in (G, k). As $G - S$ is an interval graph and so are all its induced subgraphs, therefore, we have that $S \setminus \{v\}$ is a solution to IVD in (G', k).

In the reverse direction, let S be a solution to IVD in (G', k). We will show that $G - S$ is an interval graph. Suppose not, then there must be an obstruction in $G - S$. Note that all the obstructions in $G - S$ are guaranteed to contain v, as otherwise, the obstruction is also present in $G' - S$, which contradicts that S is a solution to IVD in (G', k). This implies that $S \cup \{v\}$ is a $(k+1)$-solution for G. Recall that $\mathcal{W}$ is $(k+1)$-necessary, therefore $S \cup \{v\}$ hits $\mathcal{W}$. Since $v \notin M$ and $\mathcal{W} \subseteq 2^M$, we derive that S hits $\mathcal{W}$. But then any obstruction in $G - S$ is not covered by $\mathcal{W}$ since $v \notin M$. This together with the fact that M is a 9-redundant solution implies that for any obstruction $\mathbb{O}'$ in $G - S$ we have $|V(\mathbb{O}') \cap M| \geq 10$. Moreover, such an obstruction can either be a cycle, a †-AW, or a ‡-AW on at least 10 vertices. Among all obstructions in $G - S$ (containing v), we will proof the correctness of the lemma by carefully choosing an (available) obstruction, and in each case arriving at some contradiction. In the following, we describe the choice of the obstruction $\mathbb{O}$ in $G - S$.

1. If $G - S$ has an induced cycle Q (containing v) of length at least 4, then $\mathbb{O}$ is set to Q.

2. Otherwise, $\mathbb{O}$ is an obstruction in $G - S$ (containing v) of minimum possible size, and over all such minimum sized obstructions, $\mathbb{O}$ maximizes the number of vertices from B_i.

We will consider cases depending on which type of obstruction $\mathbb{O}$ is, and the role that v plays in $\mathbb{O}$. In the case when $\mathbb{O}$ is an induced cycle, our goal will be to obtain an obstruction not containing v in $G - S$. In all other cases, we either will obtain an obstruction not containing v, or a smaller sized obstruction, or an obstruction which has the same number of vertices as $\mathbb{O}$ but has more vertices from B_i than $\mathbb{O}$ has from B_i. In each such case this will contradict the choice of $\mathbb{O}$.

Next, we consider the cases depending on whether $\mathbb{O}$ is a cycle, a †-AW, or a ‡-AW.

$\mathbb{O}$ is a cycle

Let us first note that $|V(\mathbb{O}) \cap B_i| \leq 2$ as B_i is a clique. Let x, y be the neighbors of v in $\mathbb{O}$, and note that they lie in $M \cup \beta(\mathbb{P})$. Since $\mathbb{O}$ is not covered by $\mathcal{W}$, we have $|V(\mathbb{O}) \cap M| \geq 10$. Let $\widehat{M} = M \cap V(\mathbb{O})$, $M' \subseteq \widehat{M}$ of size 3 such that $\widehat{M} \cap \{x, y\} \subseteq M'$, and $\mathbb{F} = (M', M' \cap \{x, y\})$. Next, consider the sets $L_{\mathsf{far}} = L^{\mathbb{F},i}_{\mathsf{far}} \setminus (S \cup V(\mathbb{O}))$ and $R_{\mathsf{far}} = R^{\mathbb{F},i}_{\mathsf{far}} \setminus (S \cup V(\mathbb{O}))$. Since $|S| \leq k$, $v \notin H_i$, and B_i is a clique, therefore, $L_{\mathsf{far}}, R_{\mathsf{far}} \neq \emptyset$. Let $z \in M' \setminus \{x, y\}$, which exists since $|M'| = 3$. Now suppose that there is $v^* \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ such that $(v^*, x), (v^*, y) \in E(G)$ then we claim that we can obtain a cycle on at least four vertices not containing v in $G - S$. Since v^* fits $\mathbb{F}$, therefore

$(v^*, z) \notin E(G)$. Consider the paths P_{xz} and P_{yz} from x to z and y to z in $\mathbb{O} - \{v\}$, respectively. Furthermore, let x^* and y^* be the last vertices in P_{xz} and P_{yz} which are adjacent to v^*. Note that x^* and y^* exists since $(x, v^*), (y, v^*) \in E(G)$. But then the path from x^* to y^* in $\mathbb{O} - \{v\}$ along with v^* forms an induced cycle on at least 4 vertices in $G - S$ which does not contain v.

Next, we assume that any vertex in $L_{\mathsf{far}} \cup R_{\mathsf{far}}$ is adjacent to at most one of x, y. From Observation 5.1 (together with $(x, y) \notin E(G)$), it follows that, either $L_{\mathsf{far}} \subseteq N_G(x)$ and $R_{\mathsf{far}} \subseteq N_G(y)$, or $R_{\mathsf{far}} \subseteq N_G(x)$ and $L_{\mathsf{far}} \subseteq N_G(y)$, must hold. Suppose that $L_{\mathsf{far}} \subseteq N_G(x)$ and $R_{\mathsf{far}} \subseteq N_G(y)$ (the other case is symmetric). Consider vertices $u^* \in L_{\mathsf{far}}$ and $v^* \in R_{\mathsf{far}}$. Note that $(u^*, x), (v^*, y), (u^*, v^*) \in E(G)$ and $(u^*, y), (v^*, x), (u^*, z), (v^*, z) \notin E(G)$. Consider the paths P_{xz} and P_{yz} from x to z and y to z in $\mathbb{O} - \{v\}$, respectively. Let x^* be the last vertex in the path P_{xz} such that $N_G(x^*) \cap \{u^*, v^*\} \neq \emptyset$. Similarly, let y^* be the last vertex in the path P_{yz} such that $N_G(y^*) \cap \{u^*, v^*\} \neq \emptyset$. Let P_{x^*z} and P_{zy^*} be the paths from x^* to z and z to y^* in $\mathbb{O} - \{v\}$, respectively. Notice that $G[V(P_{x^*z}) \cup V(P_{zy^*}) \cup \{u^*, v^*\}]$ contains an induced cycle (not containing v) on at least 4 vertices.

$\mathbb{O}$ is a †-AW

Let $\mathbb{O}$ comprise of the base path $\mathsf{base}(\mathbb{O}) = (b_1, b_2, \ldots, b_z)$, non-shallow terminals t_ℓ and t_r, shallow terminal t, and center c (as in the definition in Section 2). Furthermore, let $P(\mathbb{O}) = (t_\ell, b_1, b_2, \ldots, b_z, t_r)$, and let $b_0 = t_\ell$, and $b_{z+1} = t_r$. Let $\widehat{M} = M \cap V(\mathbb{O})$, M' be a subset of $\widehat{M}$ of size 8 such that $\widehat{M} \cap \{c, t, t_\ell, t_r, b_1, b_2, b_{z-1}, b_z\} \subseteq M'$, and $\mathbb{F} = (M', M' \cap N_G(v))$. Next, we define the following sets, whose vertices will be used to either construct an obstruction not containing v, or an obstruction containing v but with (strictly) smaller size, or an obstruction with same number of vertices as $\mathbb{O}$ but containing strictly more vertices from B_i than $\mathbb{O}$ contains from B_i. Let $L_{\mathsf{far}} = L^{\mathbb{F},i}_{\mathsf{far}} \setminus (S \cup V(\mathbb{O}))$, $L_{\mathsf{cls}} = L^{\mathbb{F},i}_{\mathsf{cls}} \setminus (S \cup V(\mathbb{O}))$, $R_{\mathsf{far}} = R^{\mathbb{F},i}_{\mathsf{far}} \setminus (S \cup V(\mathbb{O}))$, and $R_{\mathsf{cls}} = R^{\mathbb{F},i}_{\mathsf{cls}} \setminus (S \cup V(\mathbb{O}))$. Notice that $|V(\mathbb{O}) \cap B_i| \leq 4$, since no obstruction contains a clique of size 5 and $G[B_i]$ is a clique. This together with the fact that $v \notin H_i$ and $|S| \leq k$ implies that $L_{\mathsf{far}}, L_{\mathsf{cls}}, R_{\mathsf{far}}, R_{\mathsf{cls}} \neq \emptyset$. Next, we consider cases depending on the role that v plays in the obstruction $\mathbb{O}$.

Suppose v is the shallow terminal. In this case, $(v, c) \in E(G)$ therefore, from Observation 5.1 one of $L_{\mathsf{far}} \subseteq N_G(c)$ or $R_{\mathsf{far}} \subseteq N_G(c)$ must hold. Consider the case when $L_{\mathsf{far}} \subseteq N_G(c)$ (the other case is symmetric), and let v^* be a vertex in L_{far}. Next, we consider the following cases based on the neighborhood of v^* in $\mathbb{O}$. (see Figure 2).

Case †-AW.S.1. $|N_G(v^*) \cap V(P(\mathbb{O}))| = 0$. In this case, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{v^*\}]$ is a †-AW in $G' - S$.

Case †-AW.S.2 If $|N_G(v^*) \cap V(P(\mathbb{O}))| = 1$. If $(v^*, t_\ell) \in E(G)$ then $G[\{v^*, c, t_\ell, b_1\}]$ is an induced cycle on 4 vertices not containing v in $G - S$. Analogous argument can be given when $(v^*, t_r) \in E(G)$. Therefore, we assume that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_i\}$, where $i \in [z]$. If $i \in [z] \setminus \{1, z\}$ then $G[\{v^*, v, b_i, b_{i-1}, b_{i-2}, b_{i+1}, b_{i+2}\}]$ is a long claw in $G - S$. This cannot happen as any obstruction in $G - S$ is of size at least 10. If none of the above cases are applicable then $N_G(v^*) \cap V(P(\mathbb{O})) \in \{\{b_1\}, \{b_z\}\}$. Suppose that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_1\}$ (the other case is symmetric) then $G[\{c, v, v^*, b_1, b_2, b_3, t_\ell\}]$ is a whipping top in $G - S$.

Case †-AW.S.3 $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2$. If neighbors of v^* are not consecutive in the path $P(\mathbb{O})$ then we can obtain an induced cycle on at least 4 vertices in $G[\{v^*\} \cup V(P(\mathbb{O}))]$, therefore we assume that the neighbors of v^* in $P(\mathbb{O})$ are consecutive. By the construction of $\mathbb{F}$ and v^* we know that there are at least 9 vertices in $P(\mathbb{O})$ which are non-adjacent to v^*. This also

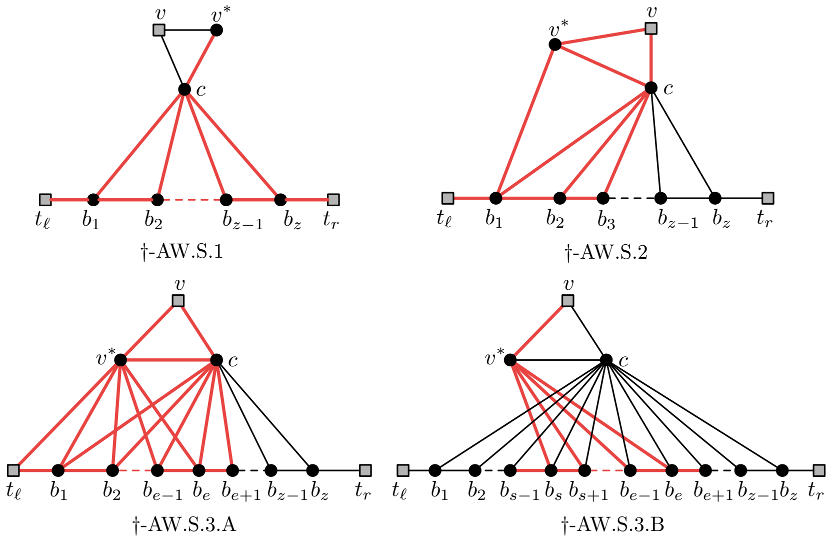

Figure 2: Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = t$.

implies that $|\{t_\ell, t_r\} \cap N_G(v^*)| \leq 1$. Without loss of generality we assume that $(v^*, t_r) \notin E(G)$. Next, we consider the following cases based on whether or not $(v^*, t_\ell) \in E(G)$.

A) $(v^*, t_\ell) \in E(G)$. In this case, there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. Let $V' = \{v, v^*, c, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW with $|V'| < |V(\mathbb{O})|$, a contradiction to the choice of $\mathbb{O}$.

B) $(v^*, t_\ell) \notin E(G)$. Let b_s and b_e be the first and the last vertices in $P(\mathbb{O})$ which are adjacent to v^*, respectively. Notice that $s \neq e$ (since $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2$), and $\{b_s, b_{s+1}, \ldots b_e, b_{e+1}\} \subset \{b_1, b_2, \ldots b_z\}$ (strict subset). Let $V' = \{v, v^*\} \cup \{b_{s-1}, b_s, b_{s+1}, \ldots, b_e, b_{e+1}\}$. Observe that $|V'| < |V(\mathbb{O})|$ and $G[V']$ is a †-AW.

Suppose v is the center. In this case, $(t_\ell, v), (t_r, v) \notin E(G)$. Since $v \notin H_i$ and each vertex in $L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ fits the frame $\mathbb{F}$, from Observation 5.2 one of the following holds. (1) $N_G(t_\ell) \cap L_{\mathsf{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\mathsf{cls}} = \emptyset$; (2) $N_G(t_r) \cap L_{\mathsf{cls}} = \emptyset$ and $N_G(t_\ell) \cap R_{\mathsf{cls}} = \emptyset$; (3) $N_G(t_\ell) \cap L_{\mathsf{cls}} = \emptyset$ and $N_G(t_r) \cap L_{\mathsf{cls}} = \emptyset$; (4) $N_G(t_\ell) \cap R_{\mathsf{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\mathsf{cls}} = \emptyset$. Consider a vertex $v^* \in L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$, and let b_s and b_e be the first and the last vertices in the path $P(\mathbb{O})$ which are adjacent to v^*, respectively. The existence and distinctness of b_s, b_e follows from the fact that $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 5$, which in turn is implied from the choice of M' and v^* fitting the frame $\mathbb{F}$. The neighbors of v^* in $P(\mathbb{O})$ must be consecutive, otherwise we can obtain an induced cycle of length at least 4, which does not contain v. We further consider sub-cases based on whether or not the following two criterions are satisfied (see Figure 3).

1. $t \in N_G(v^*)$;
2. $N_G(v^*) \cap \{t_\ell, t_r\} = \emptyset$.

Case †-AW.C.1. $t \notin N_G(v^*)$. If $\{t_\ell, t_r\} \subseteq N_G(v^*)$ then $G[\{v^*, t_\ell, b_1, v, b_z, t_r, t\}]$ is a whipping top. Here, we rely on the fact that neighbors of v^* in $P(\mathbb{O})$ are consecutive. From the above, we can assume that $|\{t_\ell, t_r\} \cap N_G(v^*)| \leq 1$. Let $V' = (V(\mathbb{O}) \setminus \{b_{s+1}, b_{s+2}, \ldots, b_{e-1}\}) \cup \{v^*\}$. Notice that $|V'| < |V(\mathbb{O})|$ since $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 7$ and neighbors of v^* are

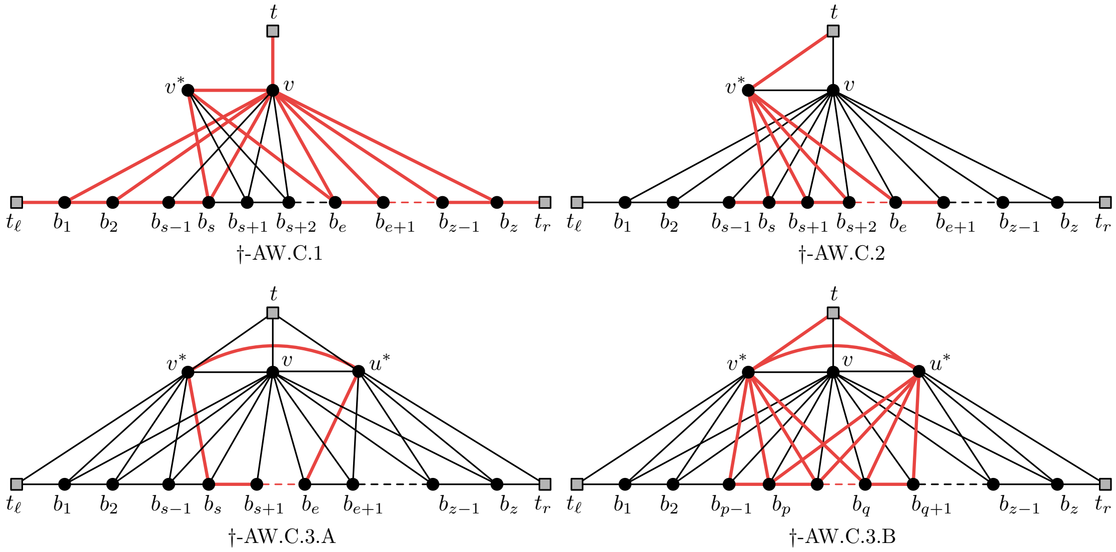

Figure 3: Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = c$.

consecutive. Moreover, $G[V']$ is an (induced) †-AW or a net, which is of strictly smaller size than $\mathbb{O}$, contradicting the choice of $\mathbb{O}$. Here, we crucially rely on the fact that $|N_G(v^*) \cap \{t_\ell, t_r\}| \leq 1$.

Case †-AW.C.2. $t \in N_G(v^*)$ and $N_G(v^*) \cap \{t_\ell, t_r\} = \emptyset$. In this case, $G[\{v^*, t, b_{s-1}, b_s, b_{s+1}, \ldots, b_e, b_{e-1}\}]$ forms an (induced) †-AW in $G - S$ which does not contain v.

If Case †-AW.C.1 and †-AW.C.2 are not applicable then for each $u \in L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ we have $t \in N_G(u)$ and $N_G(u) \cap \{t_\ell, t_r\} \neq \emptyset$. Furthermore, $v \notin H_i$, $(t_\ell, v), (t_r, v) \notin E(G)$, and each vertex in $L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ fits the frame $\mathbb{F}$. Therefore, one of the following must hold. 1) $N_G(t_\ell) \cap L_{\mathsf{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\mathsf{cls}} = \emptyset$; 2) $N_G(t_r) \cap L_{\mathsf{cls}} = \emptyset$ and $N_G(t_\ell) \cap R_{\mathsf{cls}} = \emptyset$. Thus for each $u \in L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ we have $|N_G(u) \cap \{t_\ell, t_r\}| = 1$. We assume that $N_G(t_\ell) \cap L_{\mathsf{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\mathsf{cls}} = \emptyset$ (the other case is symmetric). Next, we consider a vertex $u^* \in L_{\mathsf{cls}}$ and a vertex $v^* \in R_{\mathsf{cls}}$. Notice that (by the above discussion) $t \in N_G(u^*) \cap N_G(v^*)$, $t_\ell \notin N_G(u^*)$, $t_r \in N_G(u^*)$, $t_r \notin N_G(v^*)$, and $t_\ell \in N_G(v^*)$. Also, since $u^*, v^* \in B_i$ we have $(u^*, v^*) \in E(G)$. We now consider the remaining case.

Case †-AW.C.3. $t \in N_G(u^*) \cap N_G(v^*)$, $N_G(u^*) \cap \{t_\ell, t_r\} = \{t_r\}$, and $N_G(v^*) \cap \{t_\ell, t_r\} = \{t_\ell\}$. We consider the following sub-cases.

A) If u^* and v^* have no common neighbor in $P(\mathbb{O})$ then $G[\{u^*, v^*\} \cup V(P(\mathbb{O}))]$ contains an (induced) cycle on at least 4 vertices.

B) Otherwise, u^* and v^* have at least one common neighbor in $P(\mathbb{O})$. Let b_p and b_q be the first and the last common neighbors of u^* and v^* in $P(\mathbb{O})$, respectively. Notice that $b_{p-1} \in N_G(v^*)$ and $b_{p-1} \notin N_G(u^*)$. This follows from the fact that $t_\ell, b_q \in N_G(v^*)$, neighbors of v^* are consecutive vertices in $P(\mathbb{O})$, $t_\ell \notin N_G(u^*)$, and p is the first common neighbor of u^* and v^* in $P(\mathbb{O})$. Similarly, we can argue that $b_{q+1} \in N_G(u^*)$ and $b_{q+1} \notin N_G(v^*)$. Consider the set $V' = \{t, v^*, u^*\} \cup \{b_{p-1}, b_p, \ldots, b_q, b_{q+1}\}$. Notice that $G[V']$ is a ‡-AW or a tent which does not contain v.

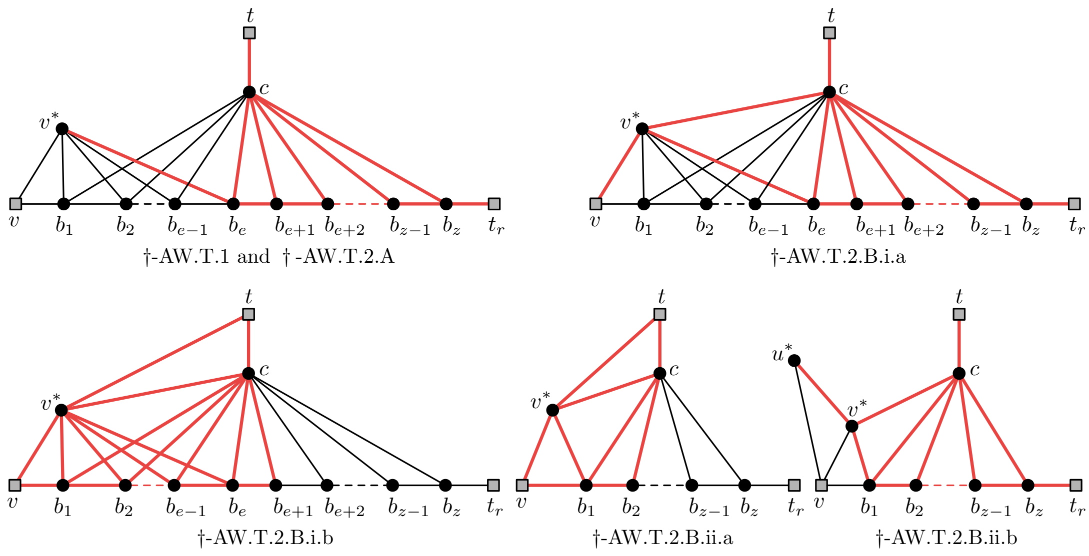

Figure 4: Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = t_\ell$.

Suppose v is one of the non-shallow terminals. We consider the case when $v = t_\ell$. By a symmetric argument we can handle the case when $v = t_r$. If $c \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ we have $(u, c) \notin E(G)$, as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P}))$. Otherwise $c \in \beta(\mathbb{P})$, and then from Observation 5.2 at least one of $L_{\mathsf{cls}} \cap N_G(c) = \emptyset$ or $R_{\mathsf{cls}} \cap N_G(c) = \emptyset$ holds. Let $X_{\mathsf{cls}} \in \{L_{\mathsf{cls}}, R_{\mathsf{cls}}\}$ be a set such that $X_{\mathsf{cls}} \cap N_G(c) = \emptyset$. Similarly, if $b_1 \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_1) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_1 \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(b_1)$ or $R_{\mathsf{far}} \subseteq N_G(b_1)$ holds (see Observation 5.1). Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_1)$. Next, we consider cases based on whether or not $b_1 \in B_i$ (see Figure 4).

Case †-AW.T.1. $b_1 \in B_i$. Consider a vertex $v^* \in X_{\mathsf{cls}}$. Note that $(v^*, b_1) \in E(G)$ since $b_1 \in B_i$, and $(v^*, c) \notin E(G)$ by the choice of v^*. Also, $(v^*, t) \notin E(G)$ otherwise, $G[\{t, c, b_1, v^*\}]$ is cycle on 4 vertices in $G - S$. Recall that v^* fits the frame $\mathbb{F}$ (and $(b_1, v^*) \in E(G)$), therefore there exists b_e such that $b_e \in N_G(u^*)$ and $b_{e+1} \notin N_G(u^*)$, where $e \in [z-1]$ (possibly $e = 1$). This together with the fact that neighbors of v^* in $P(\mathbb{O})$ are consecutive (otherwise, we obtain an induced cycle on at least 4 vertices not containing v) implies that $(v^*, t_r) \notin E(G)$. But then $G[\{t, c, v^*\} \cup \{b_e, b_{e+1}, \ldots, b_z, t_r\}]$ is a †-AW (or a net) which does not contain v.

Case †-AW.T.2. $b_1 \notin B_i$. Consider a vertex $v^* \in Y_{\mathsf{far}} \cup \{u \in X_{\mathsf{cls}} \mid (u, b_1) \in E(G)\}$, and the following cases based on its neighborhood in $\mathbb{O}$.

A) $(v^*, c) \notin E(G)$. In this case, $(v^*, t) \notin E(G)$, otherwise $G[\{v^*, t, c, b_1\}]$ is an induced cycle on 4 vertices. Recall that v^* fits the frame $\mathbb{F}$, therefore there are at least 5 vertices in $P(\mathbb{O})$ which are non-adjacent to v^*. This together with the fact that $(b_1, v^*) \in E(G)$ implies that there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. But then $G[V']$ is a †-AW (or a net) not containing v in $G - S$, where $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$.

B) $(v^*, c) \in E(G)$. We further consider the following cases.

i) There exists $e \in [z] \setminus \{1\}$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. By the choice of M' and the fact that v^* fits $\mathbb{F}$, we have $e \leq z - 2$. Consider the following cases based on whether or not $(t, v^*) \in E(G)$.

a) $(t, v^*) \notin E(G)$. Let $V' = \{t, c, v^*, v, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$. Furthermore, either $|V'| < |V(\mathbb{O})|$ or $|V'| = |V(\mathbb{O})|$ and $|V' \cap B_i| > |V(\mathbb{O}) \cap B_i|$. Here, we rely on the fact that $b_1 \notin B_i$. In either case we obtain a contradiction to the choice of $\mathbb{O}$.

b) $(t, v^*) \in E(G)$. Let $V' = \{t, c, v^*, v\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ and $|V'| < |V(\mathbb{O})|$, which contradicts the choice of $\mathbb{O}$.

ii) Otherwise, if i) does not hold then the only neighbors of v^* in $P(\mathbb{O})$ are b_1 and v. Consider the following cases based on whether or not $(t, v^*) \in E(G)$.

a) $(t, v^*) \in E(G)$. In this case, $G[\{v, v^*, t, c, b_1, b_2\}]$ is a tent.

b) $(t, v^*) \notin E(G)$. We consider a vertex in $u^* \in X_{\mathsf{cls}}$ to obtain the desired obstruction. We can assume that $(b_1, u^*) \notin E(G)$ as $X_{\mathsf{cls}} \cap N_G(c) = \emptyset$ and Case †-AW.T.2.A is not applicable. Furthermore, $(b_j, u^*) \notin E(G)$, for each $j \in [z] \setminus \{1\}$, otherwise $G[\{v, u^*\} \cup \{b_1, b_2, \ldots b_j\}]$ will contain an induced cycle on at least 4 vertices. Let $V' = (V(\mathbb{O}) \setminus \{v\}) \cup \{v^*, u^*\}$. Observe that $G[V']$ is a †-AW which does not contain v.

Suppose v is either b_1 or b_z. Suppose $v = b_1$ (the other case is symmetric). If $t_\ell \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, t_\ell) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $t_\ell \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(t_\ell)$ or $R_{\mathsf{far}} \subseteq N_G(t_\ell)$ holds (see Observation 5.1). Let $X_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $X_{\mathsf{far}} \subseteq N_G(t_\ell)$. Similarly, if $b_2 \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_2) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_2 \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(b_2)$ or $R_{\mathsf{far}} \subseteq N_G(b_2)$ holds. Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_2)$. Next, we consider cases depending on the neighborhood of vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ in $\mathbb{O}$ (see Figure 5).

Case †-AW.B.1. There is a vertex $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $\{t_\ell, b_2\} \subseteq N_G(v^*)$. There exists $e \leq z - 2$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. This follows from the fact that $(v^*, b_2) \in E(G)$ and v^* fits the frame $\mathbb{F}$. Next, we consider the sub-cases based on whether or not $(v^*, c), (v^*, t) \in E(G)$.

A) $(v^*, c) \in E(G), (v^*, t) \notin E(G)$. Let $V' = \{t, c, v^*, t_\ell, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW which does not contain v.

B) $(v^*, c) \in E(G), (v^*, t) \in E(G)$. Let $V' = \{t, c, v^*, v, t_\ell\} \cup \{b_2, b_3, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW which has strictly fewer vertices than $\mathbb{O}$.

C) $(v^*, c) \notin E(G)$. Notice that in this case $(v^*, t) \notin E(G)$, otherwise $G[\{v^*, t, c, b_2\}]$ is an induced cycle on 4 vertices. Let $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is an induced †-AW which does not contain v.

Case †-AW.B.2. Suppose that for every $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $(u, c) \in E(G)$. Since Case †-AW.B.1 is not applicable, we can assume that for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $\{t_\ell, b_2\} \not\subseteq N_G(u)$. By the construction of X_{far} and Y_{far} we know that for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $\{t_\ell, b_2\} \cap N_G(u) \neq \emptyset$, and $X_{\mathsf{far}}, Y_{\mathsf{far}} \neq \emptyset$. Consider a vertex $v^* \in X_{\mathsf{far}}$ and a vertex $u^* \in Y_{\mathsf{far}}$. We have that $(v^*, c), (u^*, c), (v^*, t_\ell), (u^*, b_2) \in E(G)$ and $(v^*, b_2), (u^*, t_\ell) \notin E(G)$. Next, we consider cases based on whether or not t adjacent to v^* and u^*.

A) $(t, v^*) \in E(G)$. Recall that $b_2 \notin N_G(v^*)$ and $t_\ell, t, c \in N_G(v^*)$. But then $G[\{c, v, v^*, b_2, t_\ell, t\}]$ is a tent in $G - S$.

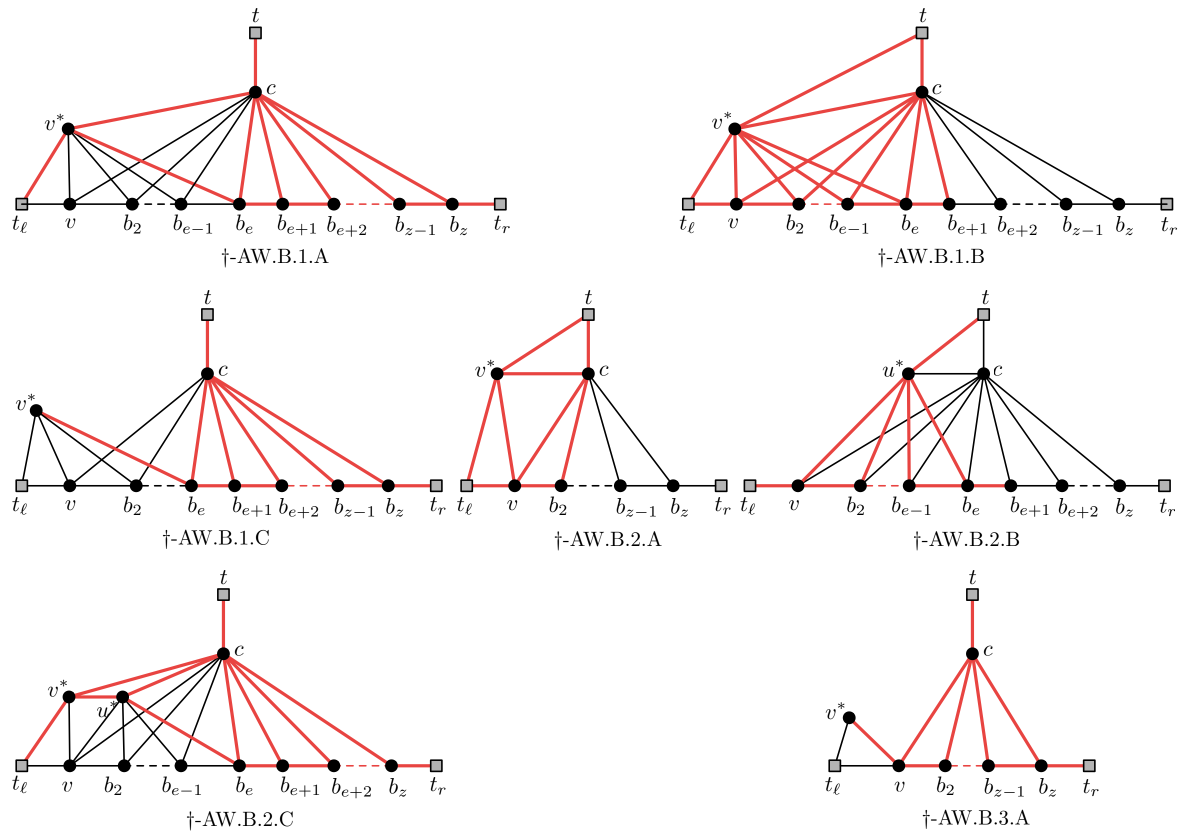

Figure 5: Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = b_1$.

B) $(t, u^*) \in E(G)$. There exists $e \in [z-2]$ such that $b_e \in N_G(u^*)$ and $b_{e+1} \notin N_G(u^*)$. This follows from the fact that $(u^*, b_2) \in E(G)$ and u^* fits the frame $\mathbb{F}$. Let $V' = \{b_2, b_3, \ldots, b_e, b_{e+1}\} \cup \{t, u^*, t_\ell, v\}$. Then $G[V']$ is a †-AW in $G - S$ which has strictly fewer vertices than $\mathbb{O}$.

C) $(t, v^*), (t, u^*) \notin E(G)$. We start by arguing that v^* cannot be adjacent to b_j, where $j \in [z] \setminus \{1\}$. For $j = 2$ it follows from the choice of v^*. Next consider the smallest $j > 2$ such that $(v^*, b_j) \in E(G)$. Then $G[\{v, v^*\} \cup \{b_2, b_3, \ldots, b_j\}]$ is an induced cycle on at least 4 vertices, which has strictly less number of vertices than $\mathbb{O}$. Therefore, we assume that the only neighbor of v^* in $P(\mathbb{O})$ are v and t_ℓ. Next, we argue about neighbors of u^* in $P(\mathbb{O})$. There exists $e \in [z-2]$ such that $b_e \in N_G(u^*)$ and $b_{e+1} \notin N_G(u^*)$. This follows from the fact that $(u^*, b_2) \in E(G)$ and u^* fits the frame $\mathbb{F}$. Let $V' = \{t, c, t_\ell, t_r, v^*, u^*\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$ which does not contain v.

Case †-AW.B.3. Suppose that there is $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(u, c) \notin E(G)$, and for all $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $\{t_\ell, b_2\} \not\subseteq N_G(u)$. Consider vertices $v^* \in X_{\mathsf{far}}$ and $u^* \in Y_{\mathsf{far}}$, and the following sub-cases.

A) Consider the case when $(v^*, c) \notin E(G)$. This implies that $(v^*, t) \notin E(G)$, otherwise $G[v^*, c, t, v]$ is a cycle on 4 vertices. As Case †-AW.B.1 is not applicable, for each $u \in Y_{\mathsf{far}}$ we have $(u, b_2) \in E(G)$ and $(u, t_\ell) \notin E(G)$. Note that since v is unmarked, therefore, $Y_{\mathsf{far}} \neq \emptyset$. From the above discussions we obtain that $t_\ell \notin B_i$. Observe that v^* cannot be adjacent to any b_j, where $j \geq 2$, since the neighbors of v^* in $P(\mathbb{O})$ must be consecutive,

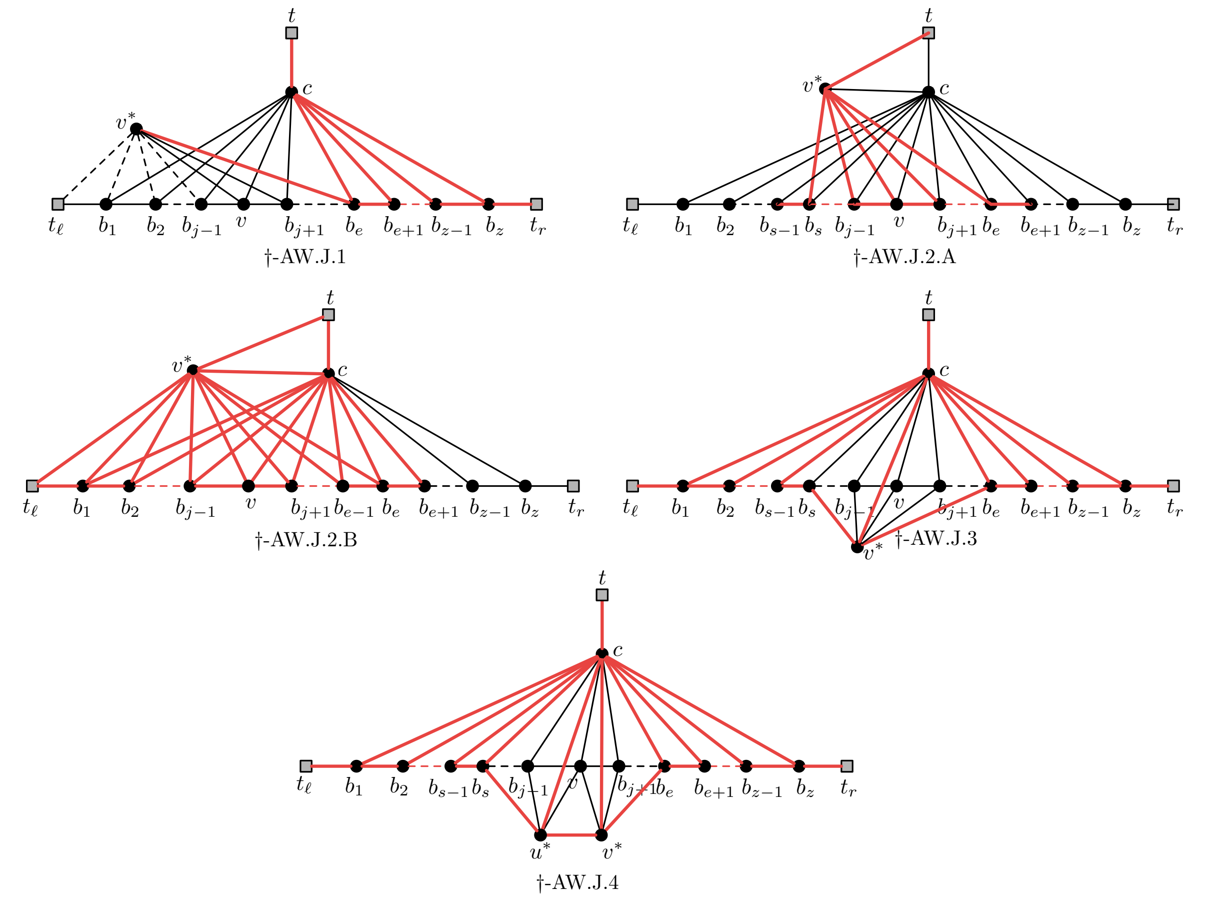

Figure 6: Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = b_j$, where $j \in [z-1] \setminus \{1\}$.

$(v^*, t_\ell) \in E(G)$, and $(v^*, b_2) \notin E(G)$. But then $G[(V(\mathbb{O}) \setminus \{t_\ell\}) \cup \{v^*\}]$ is a †-AW with same number of vertices as $\mathbb{O}$ but with more vertices from B_i.

B) Consider the case when $(u^*, c) \notin E(G)$. Since Case †-AW.B.3.A is not applicable we can assume that $(v^*, c) \in E(G)$. Observe that $G[\{c, v^*, u^*, b_2\}]$ is a cycle on 4 vertices. Here, we rely on the fact that $(v^*, b_2) \notin E(G)$.

Suppose that v is a base vertex b_j, where $j \in [z] \setminus \{1, z\}$. Let $X_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $X_{\mathsf{far}} \subseteq N_G(b_{j-1})$ and $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_{j+1})$. We note that existence of X_{far} and Y_{far} is guaranteed from Observation 5.1. Next, we consider cases based on the neighborhood of vertices in X_{far} and Y_{far} in $\mathbb{O}$ (see Figure 6).

Case †-AW.J.1. If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, c) \notin E(G)$. Note that, as $(v^*, c) \notin E(G)$, we have $(v^*, t) \notin E(G)$, otherwise $G[\{v, v^*, c, t\}]$ would be an induced cycle on 4 vertices. All the neighbors of v^* on $P(\mathbb{O})$ must be consecutive. This together with the choice of $\mathbb{F}$ and v^* implies that, one of (a) $\{t_\ell, b_1\} \cap N_G(v^*) = \emptyset$ or (b) $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ must hold. Suppose that $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ (the other case is symmetric). Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $t_r, b_z \notin N_G(v^*)$ and $N_G(v^*) \cap \{v, b_{j-1}, b_{j+1}\} \neq \emptyset$. We note that e could possibly be equal to j. Let $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $|V'| < |V(\mathbb{O})|$ since $j \in [z] \setminus \{1, z\}$. Moreover, $G[V']$ is a †-AW in $G - S$, which contradicts the choice of $\mathbb{O}$.

Note that if Case †-AW.J.1 is not applicable then for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $(u, c) \in E(G)$. Next, we consider cases based on whether or not the following conditions are satisfied for a vertex $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$.

1. $(u, t) \in E(G)$;
2. $\{b_{j-1}, b_{j+1}\} \subseteq N_G(u)$.

Case †-AW.J.2 If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, t) \in E(G)$. We start by recalling the following. Since M is a 9-redundant solution and $\mathbb{O}$ is not covered by $\mathcal{W}$, we have $|M \cap V(\mathbb{O})| \geq 10$, which implies that $|V(\mathbb{O})| \geq 10$. By the choice of $\mathbb{F}$ and the fact that $2 \leq j \leq z - 1$ (where $v = b_j$), we have at least 4 vertices in $V(P(\mathbb{O}))$ which are non-adjacent to v^*. Moreover, by our assumption that there is no obstruction which is an induced cycle on at least 4 vertices, we have that all the neighbors of v^* in $P(\mathbb{O})$ must be consecutive. From the above discussions, we can conclude that at least one of $\{b_1, b_2, t_\ell\} \cap N_G(v^*) = \emptyset$ or $\{b_{z-1}, b_z, t_r\} \cap N_G(v^*) = \emptyset$ must hold. Suppose that $\{b_{z-1}, b_z, t_r\} \cap N_G(v^*) = \emptyset$ holds (the other case is symmetric). We further consider the following sub-cases based on whether or not $t_\ell \in N_G(v^*)$.

A) $t_\ell \notin N_G(v^*)$. Let $s \in [j]$ such that b_s is the first vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $(t_\ell, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Also, let $e \in [z-2]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $(t_r, v^*), (b_z, v^*), (b_{z-1}, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Notice that $s \neq e$ since by the construction of the sets X_{far} and Y_{far} we have that v^* is incident to v and at least one of the vertices in $\{b_{j-1}, b_{j+1}\}$. Let $V' = \{t, v^*\} \cup \{b_{s-1}, b_s, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a †-AW in $G - S$. Moreover, $|V'| < |V(\mathbb{O})|$ since $t_r, c, b_z \notin V'$ and $V' \subseteq V(\mathbb{O}) \cup \{v^*\}$.

B) $t_\ell \in N_G(v^*)$. Let $e \in [z-2]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $(t_r, v^*), (b_z, v^*), (b_{z-1}, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Let $V' = \{t, v^*, c, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$. Moreover, $|V'| < |V(\mathbb{O})|$ since $t_r, b_z \notin V'$ and $V' \subseteq V(\mathbb{O}) \cup \{v^*\}$.

Case †-AW.J.3. If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, t) \notin E(G)$ and $\{b_{j-1}, b_{j+1}\} \subseteq N_G(v^*)$. Notice that all the neighbors of v^* on $P(\mathbb{O})$ must be consecutive, and there are at least 4 vertices on $P(\mathbb{O})$ that are non-adjacent to v^*. This follows from the facts that M is a 9-redundant solution, $\mathbb{O}$ is not covered by $\mathcal{W}$, $G - S$ has no obstructions which are induced cycles, and the choices of $\mathbb{F}$ and v^*. From the above discussions, we can conclude that one of $\{t_\ell, b_1\} \cap N_G(v^*) = \emptyset$ or $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ must hold. Suppose that $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ (other case is symmetric). Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $t_r, b_z \notin N_G(v^*)$ and $\{b_{j-1}, b_{j+1}\} \subseteq N_G(v^*)$. Also, let $s \in [z-1] \cup \{0\}$ be the lowest integer such that $(v^*, b_s) \in E(G)$ (b_s could possibly be same as b_{j-1} or $b_0 = t_\ell$). Let $V' = \{t, c, v^*, t_\ell, t_r\} \cup \{b_1, b_2, \ldots, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is an induced †-AW in $G - S$, which does not contain v. Here, we rely on the fact that Case †-AW.J.1 is not applicable, due to which we have $(v^*, c) \in E(G)$.

Case †-AW.J.4 For all $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $(v^*, t) \notin E(G)$, and $\{b_{j-1}, b_{j+1}\} \not\subseteq N_G(v^*)$. The non-applicability of Case †-AW.J.1, †-AW.J.2, and †-AW.J.3 (together with the constructions of X_{far} and Y_{far}) imply that, for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $(u, c) \in E(G)$, $(u, t) \notin E(G)$, and $|N_G(u) \cap \{b_{j-1}, b_{j+1}\}| = 1$. Next, consider a vertex $u^* \in X_{\mathsf{far}}$ and $v^* \in Y_{\mathsf{far}}$. Let $s \in [j-1] \cup \{0\}$ such that b_s is the first vertex in $P(\mathbb{O})$ adjacent to u^*, which exists since $(u^*, b_{j-1}) \in E(G)$. Also, let $e \in [z+1]$ such that b_e is the last vertex in $P(\mathbb{O})$ adjacent to v^*, which exists since

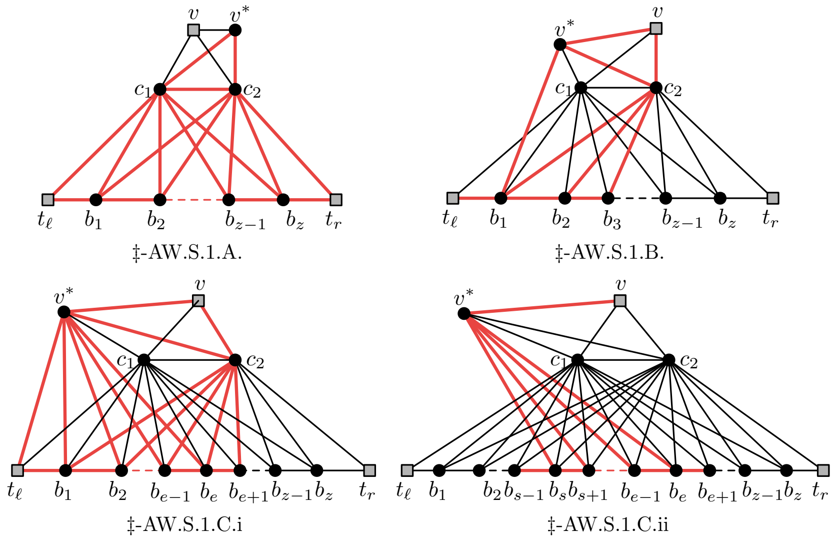

Figure 7: Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = t$.

$(v^*, b_{j+1}) \in E(G)$. Recall that $(u^*, b_{j-1}), (v^*, b_{j+1}) \in E(G)$ and $(u^*, b_{j+1}), (v^*, b_{j-1}) \notin E(G)$. Moreover, the neighbors of u^* and the neighbors of v^* in $P(\mathbb{O})$ must be consecutive vertices in $P(\mathbb{O})$, respectively. From above discussions, we can conclude that $s \neq e$. Now we let $V' = \{t, c, v^*, u^*\} \cup \{t_\ell, b_1, b_2, b_{s-1}, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW (or a net) in $G - S$ which does not contain v.

$\mathbb{O}$ is a ‡-AW

Let $\mathbb{O}$ comprise of the base path $\mathsf{base}(\mathbb{O}) = (b_1, b_2, \ldots, b_z)$, non-shallow terminals t_ℓ and t_r, shallow terminal t, and centers c_1 and c_2 (as in the definition in Section 2). Furthermore, let $P(\mathbb{O}) = (t_\ell, b_1, b_2, \ldots, b_z, t_r)$, $b_0 = t_\ell$, and $b_{z+1} = t_r$. Let $\widehat{M} = M \cap V(\mathbb{O})$, M' be a subset of $\widehat{M}$ of size 9 such that $\widehat{M} \cap \{c_1, c_2, t, t_\ell, t_r, b_1, b_2, b_{z-1}, b_z\} \subseteq M'$, and $\mathbb{F} = (M', M' \cap N_G(v))$. Next, we define the sets, the vertices from which will be used to either construct an obstruction not containing v, an obstruction containing v but with (strictly) smaller size, or and an obstruction with same number of vertices as $\mathbb{O}$ but containing more vertices from B_i. Let $L_{\mathsf{far}} = L^{\mathbb{F},i}_{\mathsf{far}} \setminus (S \cup V(\mathbb{O}))$, $L_{\mathsf{cls}} = L^{\mathbb{F},i}_{\mathsf{cls}} \setminus (S \cup V(\mathbb{O}))$, $R_{\mathsf{far}} = R^{\mathbb{F},i}_{\mathsf{far}} \setminus (S \cup V(\mathbb{O}))$, and $R_{\mathsf{cls}} = R^{\mathbb{F},i}_{\mathsf{cls}} \setminus (S \cup V(\mathbb{O}))$. Notice that $|V(\mathbb{O}) \cap B_i| \leq 4$, since no obstruction contains a clique of size 5 and $G[B_i]$ is a clique. This together with the fact that $v \notin H_i$ and $|S| \leq k$ implies that $L_{\mathsf{far}}, L_{\mathsf{cls}}, R_{\mathsf{far}}, R_{\mathsf{cls}} \neq \emptyset$. Next, we consider cases depending on the role that v plays in $\mathbb{O}$.

Suppose that v is the shallow terminal. For a vertex $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $\{c_1, c_2\} \cap N_G(u) \neq \emptyset$. This follows from Observation 5.1 and the fact that $(v, c_1), (v, c_2) \in E(G)$. Next, consider the following cases depending on the neighborhood of vertices in $L_{\mathsf{far}} \cup R_{\mathsf{far}}$ in $\mathbb{O}$.

Case ‡-AW.S.1. There is $v^* \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ such that $c_1, c_2 \in N_G(v^*)$. We further consider sub-cases based on other neighbors (if any) of v^* in $\mathbb{O}$ (see Figure 7).

A) $|N_G(v^*) \cap V(P(\mathbb{O}))| = 0$. In this case, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{v^*\}]$ is a ‡-AW in $G - S$.

B) If $|N_G(v^*) \cap V(P(\mathbb{O}))| = 1$. If $(v^*, t_\ell) \in E(G)$ then $G[\{v^*, c_2, t_\ell, b_1\}]$ is an induced cycle on 4 vertices. Analogous argument can be given when $(v^*, t_r) \in E(G)$. Therefore, we assume that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_i\}$, where $i \in [z]$. If $i \in [z] \setminus \{1, z\}$ then

$G[\{v^*, v, b_i, b_{i-1}, b_{i-2}, b_{i+1}, b_{i+2}\}]$ is a long claw in $G-S$. If none of the above cases are applicable, then $N_G(v^*) \cap V(P(\mathbb{O}))$ is either $\{b_1\}$ or $\{b_z\}$. Suppose that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_1\}$ (the other case is symmetric) then $G[\{c_2, v, v^*, b_1, b_2, b_3, t_\ell\}]$ is a whipping top in $G - S$.

C) $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2$. If neighbors of v^* are not consecutive in the path $P(\mathbb{O})$ then we can obtain an induced cycle on at least 4 vertices in $G[\{v^*\} \cup V(P(\mathbb{O}))]$, therefore we assume that the neighbors of v^* in $P(\mathbb{O})$ are consecutive. By the construction of $\mathbb{F}$ and v^* we know that there are at least 7 vertices in $P(\mathbb{O})$ which are non-adjacent to v^*. From the above discussions, we can conclude that $|\{t_\ell, t_r\} \cap N_G(v^*)| \leq 1$. Assume that $(v^*, t_r) \notin E(G)$ (the other case is symmetric). Next, we consider the following cases based on whether or not $(v^*, t_\ell) \in E(G)$.

i) $(v^*, t_\ell) \in E(G)$. In this case, there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. Let $V' = \{v, v^*, c_2, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW with $|V'| < |V(\mathbb{O})|$.

ii) $(v^*, t_\ell) \notin E(G)$. Let b_s and b_e be the first and the last vertex in $P(\mathbb{O})$ which are adjacent to v^*, respectively. Notice that $s \neq e$ (since $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2$), and $\{b_s, b_{s+1}, \cdots b_e, b_{e+1}\} \subset \{b_1, b_2, \ldots b_z\}$. Let $V' = \{v, v^*\} \cup \{b_{s-1}, b_s, b_{s+1}, \ldots, b_e, b_{e+1}\}$. Observe that $|V'| < |V(\mathbb{O})|$, and $G[V']$ is a †-AW.

Case ‡-AW.S.2. For all $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $|\{c_1, c_2\} \cap N_G(v^*)| = 1$. From Observation 5.1, we know that for each $c' \in \{c_1, c_2\}$, we have that one of $L_{\mathsf{far}} \subseteq N_G(c')$ or $R_{\mathsf{far}} \subseteq N_G(c')$ holds. Moreover, from our assumption that for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $|\{c_1, c_2\} \cap N_G(v^*)| = 1$, it cannot be the case that $L_{\mathsf{far}} \subseteq N_G(c_1)$ and $L_{\mathsf{far}} \subseteq N_G(c_2)$. Similarly, it cannot be the case that $R_{\mathsf{far}} \subseteq N_G(c_1)$ and $R_{\mathsf{far}} \subseteq N_G(c_2)$. From the above discussions, we can conclude that one of $L_{\mathsf{far}} \subseteq N_G(c_1)$ and $R_{\mathsf{far}} \subseteq N_G(c_2)$, or $R_{\mathsf{far}} \subseteq N_G(c_1)$ and $L_{\mathsf{far}} \subseteq N_G(c_2)$ holds. Suppose $L_{\mathsf{far}} \subseteq N_G(c_1)$ and $R_{\mathsf{far}} \subseteq N_G(c_2)$ (the other case is symmetric). Next, consider a vertex $u^* \in L_{\mathsf{far}}$ and a vertex $v^* \in R_{\mathsf{far}}$. By our assumption and non-applicability of Case ‡-AW.S.1, we have $(u^*, c_1), (v^*, c_2) \in E(G)$ and $(u^*, c_2), (v^*, c_1) \notin E(G)$. Moreover, $u^*, v^* \in B_i$ therefore, $(u^*, v^*) \in E(G)$. But then $G[\{u^*, v^*, c_1, c_2\}]$ is an induced cycle on 4 vertices.

Suppose v is one of the centers. Suppose $v = c_1$ (the other case is symmetric). From Observation 5.2, we know that at least one of $N_G(t_r) \cap L_{\mathsf{cls}} = \emptyset$ or $N_G(t_r) \cap R_{\mathsf{cls}} = \emptyset$ holds. Let $X_{\mathsf{cls}} \in \{L_{\mathsf{cls}}, R_{\mathsf{cls}}\}$ be a set such that $N_G(t_r) \cap X_{\mathsf{cls}} = \emptyset$. Consider a vertex $v^* \in X_{\mathsf{cls}}$, and let b_s and b_e be the first and last vertex in the path $P(\mathbb{O})$ which are adjacent to v^*, respectively. Since, M is a 9-redundant solution and $\mathbb{O}$ is not covered by $\mathcal{W}$, we have that $|M \cap V(\mathbb{O})| \geq 10$. This together with the choice of $\mathbb{F}$ and v^*, and the fact that $V(P(\mathbb{O})) \setminus \{t_\ell, t_r\} \subseteq N_G(v)$, implies that b_s and b_e exist and are distinct. Moreover, from the above we can also conclude that $|N_G(v^*) \cap V(\mathsf{base}(\mathbb{O}))| \geq 4$. We also note that $e \leq z$ since $(v^*, t_r) \notin E(G)$. The neighbors of v^* in $P(\mathbb{O})$ must be consecutive, otherwise we can obtain an induced cycle of length at least 4 which does not contain v. We further consider sub-cases based on whether or not $t, c_2 \in N_G(v^*)$ (see Figure 8).

Case ‡-AW.C.1. $t, c_2 \notin N_G(v^*)$. Let $V' = \{v^*, v, c_2, t, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Notice that $|V'| < |V(\mathbb{O})|$ since $|N_G(v^*) \cap V(\mathsf{base}(\mathbb{O}))| \geq 4$ and neighbors of v^* are consecutive. Moreover, $G[V']$ is a ‡-AW or a tent, which is of strictly smaller size than $\mathbb{O}$, contradicting the choice of $\mathbb{O}$. Here, we crucially rely on the fact that $t_r \notin N_G(v^*)$.

Case ‡-AW.C.2. $t \notin N_G(v^*)$ and $c_2 \in N_G(v^*)$. Let $V' = (V(\mathbb{O}) \setminus \{b_{s+1}, b_{s+2}, \ldots, b_{e-2}, b_{e-1}\}) \cup \{v^*\}$. Notice that $|V'| < |V(\mathbb{O})|$ (since $|N_G(v^*) \cap V(\mathsf{base}(\mathbb{O}))| \geq 4$) and $G[V']$ is a ‡-AW.

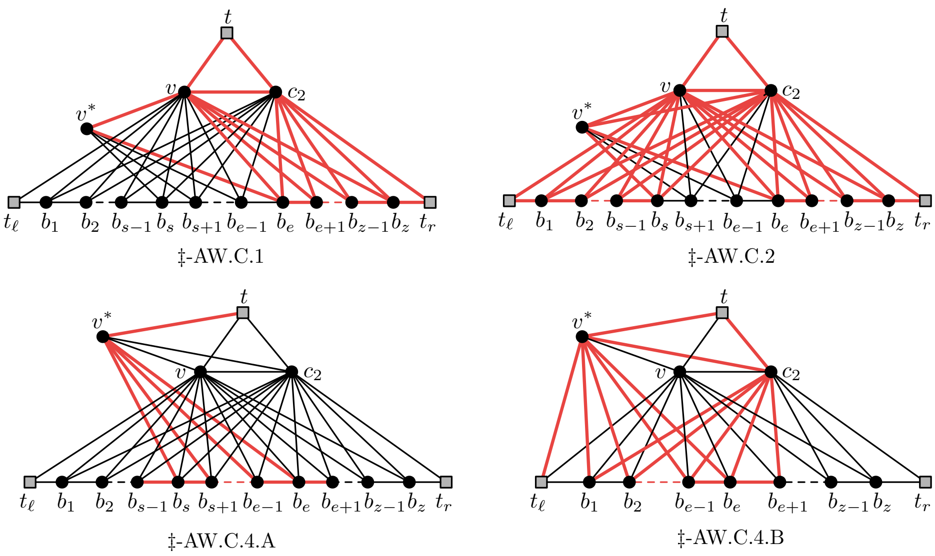

Figure 8: Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = c_1$.

Case ‡-AW.C.3. $t \in N_G(v^*)$ and $c_2 \notin N_G(v^*)$. Recall that $N_G(v^*) \cap \{b_1, b_2, \dots, b_z\} \neq \emptyset$. Consider a vertex $b_j \in N_G(v^*) \cap \{b_1, b_2, \dots, b_z\}$. The graph $G[\{v^*, t, c_2, b_j\}]$ is an induced cycle on 4 vertices.

Case ‡-AW.C.4. $t \in N_G(v^*)$ and $c_2 \in N_G(v^*)$. We further consider the following sub-cases based on whether or not $(t_\ell, v^*) \in E(G)$.

A) $(t_\ell, v^*) \notin E(G)$. Let $V' = \{t, v^*, t_\ell\} \cup \{b_{s-1}, b_s, \dots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a †-AW in $G - S$ which does not contain v.

B) $(t_\ell, v^*) \in E(G)$. Let $V' = \{t, v^*, c_2, t_\ell\} \cup \{b_1, b_2, \dots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ which does not contain v.

Suppose v is one of the non-shallow terminals. We consider the case when $v = t_\ell$. By a symmetric argument we can handle the case when $v = t_r$. If $c_2 \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ we have $(u, c_2) \notin E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $c_2 \in \beta(\mathbb{P})$, and then using Observation 5.2 we obtain that at least one of $L_{\mathsf{cls}} \cap N_G(c_2) = \emptyset$ or $R_{\mathsf{cls}} \cap N_G(c_2) = \emptyset$ holds. Let $X_{\mathsf{cls}} \in \{L_{\mathsf{cls}}, R_{\mathsf{cls}}\}$ be a set such that $X_{\mathsf{cls}} \cap N_G(c_2) = \emptyset$. Similarly, if $b_1 \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_1) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_1 \in \beta(\mathbb{P})$, and then using Observation 5.1 we obtain that at least one of $L_{\mathsf{far}} \subseteq N_G(b_1)$ or $R_{\mathsf{far}} \subseteq N_G(b_1)$ holds. Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_1)$. Next, we consider cases based on whether or not $b_1 \in B_i$ (see Figure 9).

Case ‡-AW.T.1. $b_1 \in B_i$. Consider a vertex $v^* \in X_{\mathsf{cls}}$. Note that $(b_1, v^*) \in E(G)$ since $b_1 \in B_i$, and $(v^*, c_2) \notin E(G)$, by the choice of v^*. Also, $(v^*, t) \notin E(G)$ otherwise, $G[\{t, c_2, b_1, v^*\}]$ is an induced cycle on 4 vertices in $G - S$. Recall that v^* fits the frame $\mathbb{F}$ (and $(b_1, v^*) \in E(G)$), therefore there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. This together with the fact that neighbors of v^* in $P(\mathbb{O})$ are consecutive (otherwise, we obtain an induced cycle on at least 4 vertices not containing v) implies that $(v^*, t_r) \notin E(G)$. Next, we consider cases based on whether or not $(v^*, c_1) \in E(G)$.

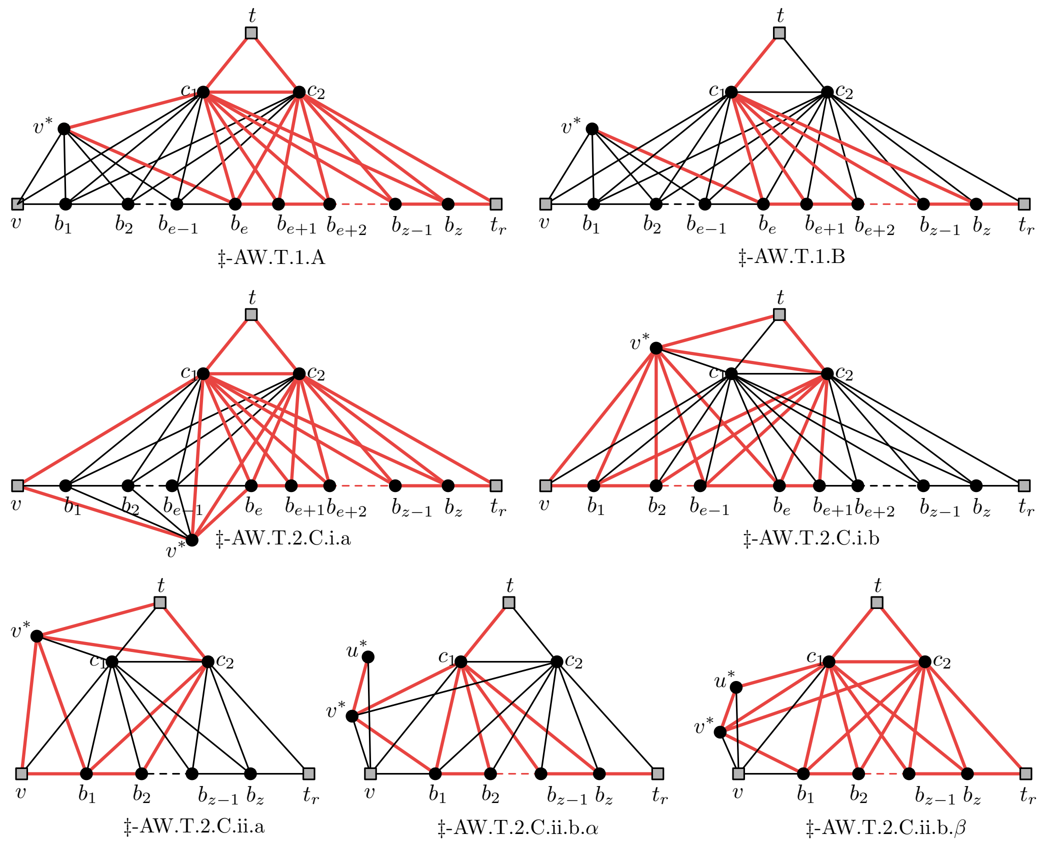

Figure 9: Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = t_\ell$.

A) $(v^*, c_1) \in E(G)$. Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ not containing v.

B) $(v^*, c_1) \notin E(G)$. Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$ not containing v.

Case ‡-AW.T.2. $b_1 \notin B_i$. Consider a vertex $v^* \in Y_{\mathsf{far}} \cup \{u \in X_{\mathsf{cls}} \mid (u, b_1) \in E(G)\}$, and the following cases based on its neighborhood in $\mathbb{O}$.

A) $(v^*, c_2) \notin E(G)$. Notice that this case is the same as Case ‡-AW.T.1, therefore we can obtain an obstruction in a similar way.

B) $(v^*, c_1) \notin E(G)$. Observe that $(v^*, t) \notin E(G)$, otherwise $G[\{v^*, b_1, c_1, t\}]$ is an induced cycle on 4 vertices in $G - S$. Now, we can obtain an obstruction as in Case ‡-AW.T.1.B.

C) $(v^*, c_1), (v^*, c_2) \in E(G)$. We further consider the following cases based on the neighborhood of v^* in $P(\mathbb{O})$.

i) There exists $e \in [z] \setminus \{1\}$ such that $(v^*, b_e) \in N_G(v^*)$ and $(v^*, b_{e+1}) \notin N_G(v^*)$. Observe that by the choices of $\mathbb{F}$ and v^*, we have $e < z - 1$. Consider the following cases based on whether or not $(t, v^*) \in E(G)$.

a) $(t, v^*) \notin E(G)$. Let $V' = \{t, c_1, c_2, v^*, v, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$. Furthermore, either $|V'| < |V(\mathbb{O})|$ or $|V'| = |V(\mathbb{O})|$ and

$|V' \cap B_i| > |V(\mathbb{O}) \cap B_i|$. Here, we rely on the fact that $b_1 \notin B_i$. In either case we obtain a contradiction to the choice of $\mathbb{O}$.

b) $(t, v^*) \in E(G)$. Let $V' = \{t, v^*, c_2, v\} \cup \{b_1, b_2, \dots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a $\ddagger$-AW in $G - S$ and $|V'| < |V(\mathbb{O})|$.

ii) If i) does not hold then the only neighbors of v^* in $P(\mathbb{O})$ are b_1 and v. Consider the following cases based on whether or not $(t, v^*) \in E(G)$.

a) $(t, v^*) \in E(G)$. In this case, $G[\{v, v^*, t, c_2, b_1, b_2\}]$ is a tent.

b) $(t, v^*) \notin E(G)$. We consider a vertex $u^* \in X_{\mathsf{cls}}$ to obtain the desired obstruction. Recall that from the construction of X_{cls} we have $(u^*, c_2) \notin E(G)$. Moreover, by the premise of Case $\ddagger$-AW.T.2.C we have $(v^*, c_2) \in E(G)$. From the above discussions, we can conclude that $(u^*, t) \notin E(G)$, as otherwise $G[\{u^*, v^*, c_2, t\}]$ is an induced cycle on 4 vertices. We assume that $(u^*, b_1) \notin E(G)$, otherwise u^* would satisfy the premise of Case $\ddagger$-AW.T.2.A and we can obtain an obstruction using it. Also, $(u^*, b_j) \notin E(G)$, for each $j \in [z] \setminus \{1\}$, otherwise $G[\{v, u^*\} \cup \{b_1, b_2, \cdots b_j\}]$ will contain an induced cycle on at least 4 vertices, which is an obstruction containing v with strictly less number of vertices than $\mathbb{O}$. Next, we consider the following cases depending on whether or not $(u^*, c_1) \in E(G)$.

α) $(u^*, c_1) \notin E(G)$. Let $V' = \{t, c_1, u^*, v^*, t_r\} \cup \{b_1, b_2, \dots, b_z\}$. Observe that $G[V']$ is a $\dagger$-AW in $G - S$, which does not contain v.

β) $(u^*, c_1) \in E(G)$. Let $V' = \{t, c_1, c_2, u^*, v^*, t_r\} \cup \{b_1, b_2, \dots, b_z\}$. Observe that $G[V']$ is a $\ddagger$-AW in $G - S$, which does not contain v.

Suppose v is b_1 or b_z. Suppose $v = b_1$ (the other case is symmetric). If $t_\ell \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, t_\ell) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $t_\ell \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(t_\ell)$ or $R_{\mathsf{far}} \subseteq N_G(t_\ell)$ holds (see Observation 5.1). Let $X_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $X_{\mathsf{far}} \subseteq N_G(t_\ell)$. Similarly, if $b_2 \notin \beta(\mathbb{P})$ then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_2) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_2 \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(b_2)$ or $R_{\mathsf{far}} \subseteq N_G(b_2)$ holds (see Observation 5.1). Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_2)$. Next, we consider cases depending on the neighborhood of vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ in $\mathbb{O}$ (see Figure 10).

Case $\ddagger$-AW.B.1. There is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $\{t_\ell, b_2\} \subseteq N_G(v^*)$. There exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. This follows from the choices of $\mathbb{F}$ and v^*, and the facts that $(v^*, b_2) \in E(G)$ and v^* fits $\mathbb{F}$. We assume that the neighbors of v^* in $P(\mathbb{O})$ are consecutive, as otherwise, we can obtain an obstruction which is an induced cycle on at least 4 vertices. Next, we consider the sub-cases based on whether or not $(v^*, c_1), (v^*, c_2), (v^*, t) \in E(G)$.

A) $(v^*, c_2) \in E(G), (v^*, t) \in E(G)$. Let $V' = \{t, c_2, v^*, t_\ell\} \cup \{b_1, b_2, \cdots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a $\ddagger$-AW such that $|V'| < |V(\mathbb{O})|$.

If Case $\ddagger$-AW.B.1.A is not-applicable then $(v^*, c_2) \notin E(G)$ or $(v^*, t) \notin E(G)$ must hold.

B) $(v^*, t) \notin E(G)$. We consider the following cases.

i) $(v^*, c_1) \notin E(G)$. Let $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \dots b_z\}$. Observe that $G[V']$ is a $\dagger$-AW in $G - S$ not containing v.

ii) $(v^*, c_1) \in E(G)$. Let $V' = \{t, c_1, c_2, v^*, t_r, t_\ell\} \cup \{b_e, b_{e+1} \dots b_z\}$. Observe that $G[V']$ contains a $\ddagger$-AW not containing v, that is present in $G - S$. We note that such an obstruction can be found both when $(v^*, c_2) \in E(G)$ and when $(v^*, c_2) \notin E(G)$.

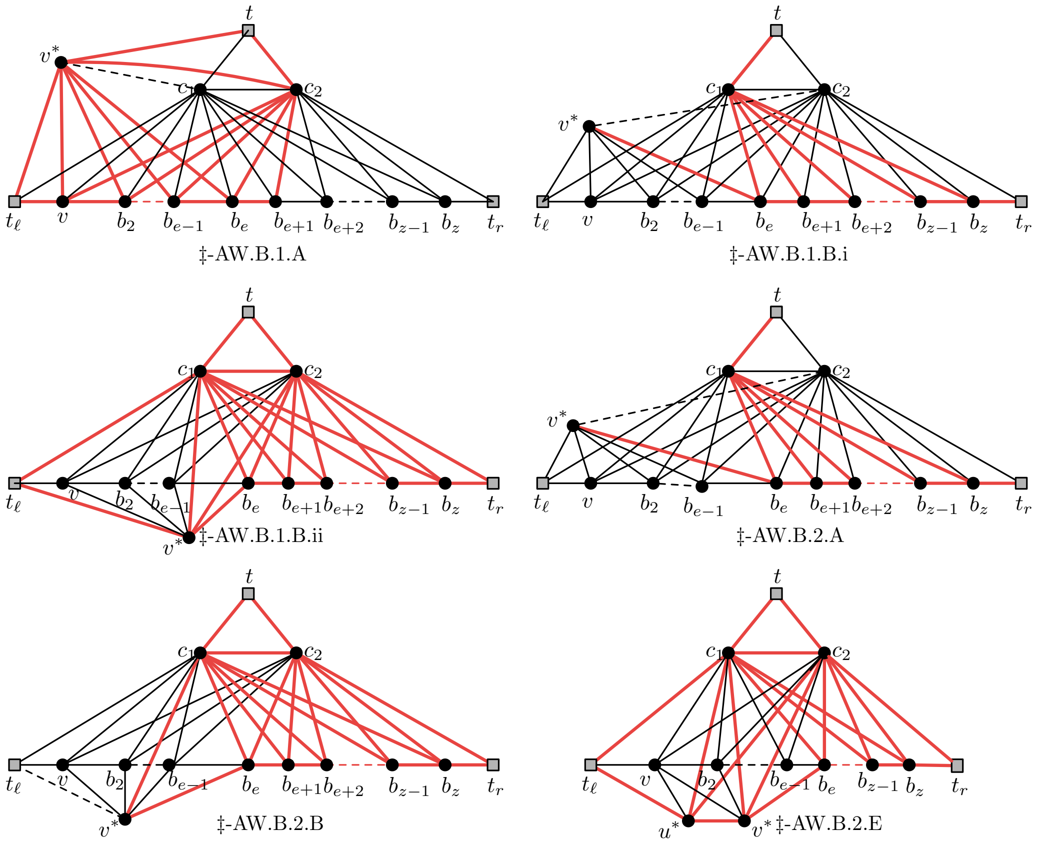

Figure 10: Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = b_1$.

C) $(v^*, c_2) \notin E(G)$. Since Case ‡-AW.B.1.B is not applicable we can assume that $(v^*, t) \in E(G)$. But then $G[\{v^*, b_2, c_2, t\}]$ is a cycle on 4 vertices.

Case ‡-AW.B.2. For all $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $\{t_\ell, b_2\} \not\subseteq N_G(u)$. Furthermore, by the construction of X_{far} and Y_{far} we know that $X_{\mathsf{far}} \subseteq N_G(t_\ell)$, $Y_{\mathsf{far}} \subseteq N_G(b_2)$, and $X_{\mathsf{far}}, Y_{\mathsf{far}} \neq \emptyset$. Hence, for any pair of vertices $u^* \in X_{\mathsf{far}}$ and $v^* \in Y_{\mathsf{far}}$, we have that $(u^*, t_\ell), (v^*, b_2) \in E(G)$ and $(u^*, b_2), (v^*, t_\ell) \notin E(G)$ (since Case ‡-AW.B.1 is not applicable). Next, we consider cases based on whether or not t and c_2 are adjacent to vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$.

A) Consider the case when there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, c_1) \notin E(G)$. In this case, $(v^*, t) \notin E(G)$, otherwise we obtain an induced cycle $G[\{v^*, v, c_1, t\}]$ on 4 vertices. Let $e \in [z-2]$ such that b_e is the last vertex in $\mathsf{base}(\mathbb{O})$ that is adjacent to v^*. Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Notice that $G[V']$ is a †-AW that has fewer vertices than $\mathbb{O}$, as we (at least) exclude c_2 and t_ℓ and include v^*.

Hereafter, we assume that for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $(u, c_1) \in E(G)$.

B) Consider the case when there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, c_2) \notin E(G)$. In this case, $(v^*, t) \notin E(G)$, otherwise $G[v^*, t, c_2, v]$ is a cycle on 4 vertices. Let $e \in [z-2]$ such that b_e is the last vertex in $\mathsf{base}(\mathbb{O})$ that is adjacent to v^*. Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Notice that $G[V']$ is a ‡-AW that has either fewer vertices than $\mathbb{O}$ or has same number of vertices as $\mathbb{O}$ but has more vertices from B_i (than $\mathbb{O}$ has from B_i). Here, we rely on the fact that $t_\ell \notin B_i$, which is ensured by the fact that $Y_{\mathsf{far}} \neq \emptyset$ and $Y_{\mathsf{far}} \cap N_G(t_\ell) = \emptyset$.

Hereafter, we will assume that for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $c_1, c_2 \in N_G(u)$.

C) If there is $u^* \in X_{\mathsf{far}}$ such that $(u^*, t) \in E(G)$. Recall that, $(u^*, t_\ell) \in E(G)$ and $(u^*, b_2) \notin E(G)$. In this case, $G[\{t, u^*, c_2, t_\ell, v, b_2\}]$ is a tent.

D) If there is $v^* \in Y_{\mathsf{far}}$ such that $(v^*, t) \in E(G)$. Recall that, $(v^*, b_2) \in E(G)$ and $(v^*, t_\ell) \notin E(G)$. Let $e \in [z-2]$ such that b_e is the last vertex in $\mathsf{base}(\mathbb{O})$ that is adjacent to v^*. Note that $e \geq 2$ as $v^* \in Y_{\mathsf{far}} \subseteq N_G(b_2)$. Let $V' = \{t, v^*, t_\ell, b_{e+1}\} \cup \{v, b_3, \dots, b_e\}$. Observe that $G[V']$ is a †-AW in $G - S$ with strictly fewer vertices than $\mathbb{O}$, as we (at least) exclude c_1, c_2 and include v^*.

E) Consider a vertex $u^* \in X_{\mathsf{far}}$ and a vertex $v^* \in Y_{\mathsf{far}}$. Since all the previous cases are not applicable, therefore $(u^*, c_1), (u^*, c_2), (v^*, c_1), (v^*, c_2) \in E(G)$, and $(u^*, t), (v^*, t) \notin E(G)$. Recall that neighbors of u^*, v^* in $P(\mathbb{O})$ are consecutive. Furthermore, $(v^*, t_\ell) \notin E(G)$ and there is no b_j adjacent to u^*, where $j \geq 2$. Let $e \in [z-2]$ such that b_e is the last neighbor of v^* in $P(\mathbb{O})$. Now, let $V' = \{t_\ell, u^*, v^*, c_1, c_2, t\} \cup \{b_e, b_{e+1}, \cdots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ which does not contain v.

Suppose $v = b_j$, where $j \in [z] \setminus \{1, z\}$. Let $X_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $X_{\mathsf{far}} \subseteq N_G(b_{j-1})$ and $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_{j+1})$. The existence of X_{far} and Y_{far} is guaranteed from Observation 5.1. Recall that $|M'| = 9$. Thus, $|V(P(\mathbb{O})) \cap M'| \geq 6$, and therefore, v must have at least 4 non-neighbors in $V(P(\mathbb{O})) \cap M'$. From the above we can conclude that one of $|(\{t_\ell\} \cup \{b_1, b_2, \dots, b_{j-2}\}) \cap (M' \setminus N_G(v))| \geq 2$ or $|(\{t_r\} \cup \{b_{j+2}, b_{j+3}, \dots, b_z\}) \cap (M' \setminus N_G(v))| \geq 2$ holds. Assume that $|(\{t_r\} \cup \{b_{j+2}, b_{j+3}, \dots, b_z\}) \cap (M' \setminus N_G(v))| \geq 2$ holds (the other case is symmetric). For each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$, the neighbors of u in $P(\mathbb{O})$ must be consecutive, otherwise, we can obtain an induced cycle on at least 4 vertices. From the above discussions, together with the facts that $(v^*, v) \in E(G)$ and v^* fits $\mathbb{F}$, we can conclude that $\{t_r, b_z\} \cap N_G(u) = \emptyset$. Here, we rely on our assumption that $|(\{t_r\} \cup \{b_{j+2}, b_{j+3}, \dots, b_z\}) \cap (M' \setminus N_G(v))| \geq 2$. We consider cases based on the neighborhood of vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ in $\mathbb{O}$ (see Figure 11).

Case ‡-AW.J.1. If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, c_1) \notin E(G)$. Note that if $(v^*, c_1) \notin E(G)$ then $(v^*, t) \notin E(G)$, otherwise $G[\{v, v^*, c_1, t\}]$ is a cycle on 4 vertices. Also, the neighbors of v^* in $P(\mathbb{O})$ must be consecutive, otherwise, we can obtain an induced cycle on at least 4 vertices. Since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v, v^*) \in E(G)$, there exists $e \in [z-1]$, such that b_e is the last vertex in $P(\mathbb{O})$ which adjacent to v^*. Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \dots, b_z\}$. Observe that $G[V']$ is a †-AW with strictly fewer vertices than $\mathbb{O}$, as we (at least) exclude c_2, t_ℓ, b_1 and include v^*.

Case ‡-AW.J.2. If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, c_2) \notin E(G)$. Since Case ‡-AW.J.1 is not applicable we can assume that $(v^*, c_1) \in E(G)$. Note that if $(v^*, c_2) \notin E(G)$ then $(v^*, t) \notin E(G)$, otherwise $G[\{v, v^*, c_2, t\}]$ is a cycle on 4 vertices. Also, the neighbors of v^* in $P(\mathbb{O})$ must be consecutive. Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v, v^*) \in E(G)$. Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_e, b_{e+1}, \dots, b_z\}$. Observe that $G[V']$ is a ‡-AW (or a net) with strictly fewer vertices than $\mathbb{O}$, as we (at least) exclude t_ℓ, b_1 and include v^*.

Note that if Cases ‡-AW.J.1 and ‡-AW.J.2 are not applicable then for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $(u, c_1), (u, c_2) \in E(G)$. Moreover, by our assumption we have $N_G(u) \cap \{t_r, b_z\} = \emptyset$. The

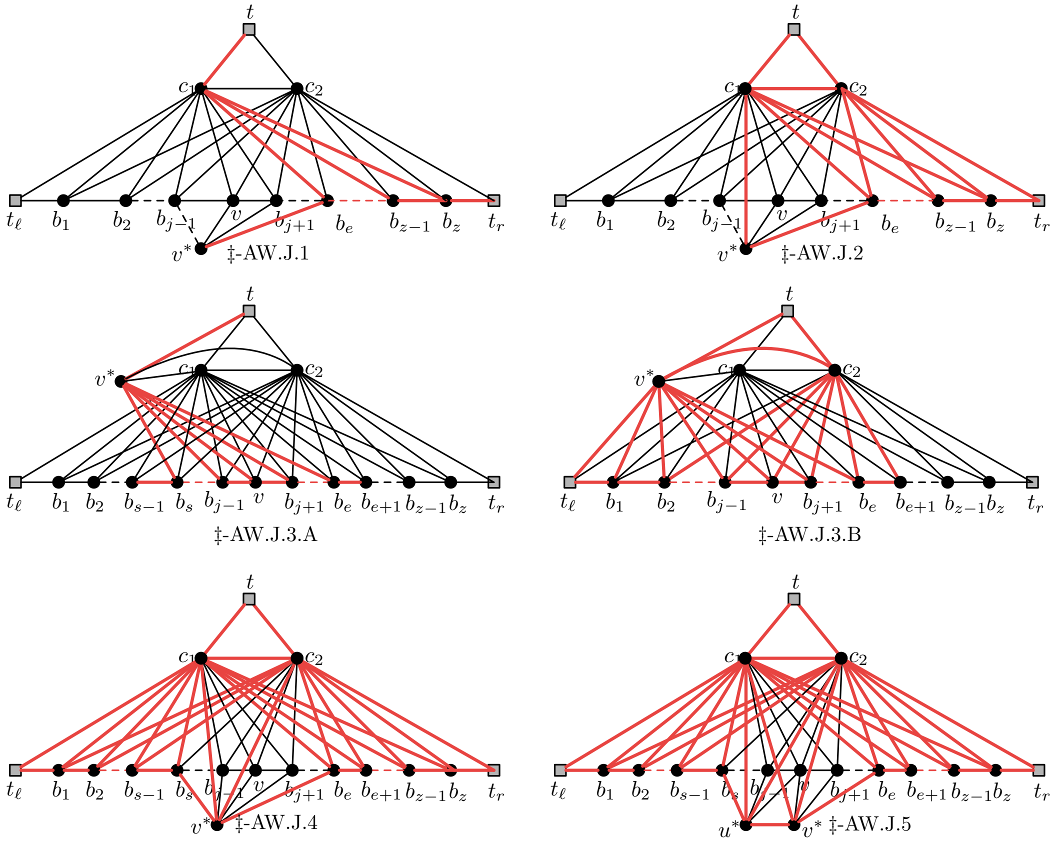

Figure 11: Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = b_j$, where $j \in [z-1] \setminus \{1\}$.

cases we consider next are based on whether or not the following conditions are satisfied for a vertex $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$.

1. $(u, t) \in E(G)$;
2. $\{b_{j-1}, b_{j+1}\} \subseteq N_G(u)$.

Case ‡-AW.J.3. If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, t) \in E(G)$. We further consider the following sub-cases based on whether or not $t_\ell \in N_G(v^*)$.

A) $t_\ell \notin N_G(v^*)$. Let $s \in [j]$ such that b_s is the first vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $(t_\ell, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Also, let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v^*, v) \in E(G)$. Notice that $s \neq e$ since by the construction of the sets X_{far} and Y_{far} we have that v^* is incident to v and at least one of the vertices in $\{b_{j-1}, b_{j+1}\}$. Let $V' = \{t, v^*\} \cup \{b_{s-1}, b_s, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a †-AW in $G - S$ with $|V'| < |V(\mathbb{O})|$. Here, we rely on the fact that $e \leq z - 1$.

B) $t_\ell \in N_G(v^*)$. Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v^*, v) \in E(G)$. Let $V' = \{t, v^*, c_2, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$ is a ‡-AW in $G - S$. Moreover, $|V'| < |V(\mathbb{O})|$ since $t_r, c_1 \notin V'$ and $V' \subseteq V(\mathbb{O}) \cup \{v^*\}$.

Case †-AW.J.4. If there is $v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ such that $(v^*, t) \notin E(G)$ and $\{b_{j-1}, b_{j+1}\} \subseteq N_G(v^*)$. Notice that all the neighbors of v^* on $P(\mathbb{O})$ must be consecutive. Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v^*, v) \in E(G)$. Also, let $s \in [z-1] \cup \{0\}$ be the lowest integer such that $(v^*, b_s) \in E(G)$ (b_s could possibly be same as b_{j-1} or $b_0 = t_\ell$). Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_1, b_2, \ldots, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ which does not contain v.

Case †-AW.J.5. For all $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $c_1, c_2 \in N_G(u)$, $(u, t) \notin E(G)$, and $\{b_{j-1}, b_{j+1}\} \not\subseteq N_G(u)$. Also, we have $X_{\mathsf{far}} \subseteq N_G(b_{j-1})$ and $Y_{\mathsf{far}} \subseteq N_G(b_{j+1})$. Next, consider a vertex $u^* \in X_{\mathsf{far}}$ and a vertex $v^* \in Y_{\mathsf{far}}$. Let $s \in [j-1] \cup \{0\}$ such that b_s is the first vertex in $P(\mathbb{O})$ which is adjacent to u^*, which exists since $(u^*, b_{j-1}) \in E(G)$. Also, let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ which is adjacent to v^*, which exists since $(t_r, v^*), (b_z, v^*) \notin E(G)$ and $(v^*, b_{j+1}) \in E(G)$. Notice that $s \neq e$. Let $V' = \{t, c_1, c_2, v^*, u^*\} \cup \{t_\ell, b_1, b_2, b_{s-1}, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ which does not contain v.

We have exhaustively considered all the cases, and obtained a desired type of obstruction for each of the cases. This concludes the proof of Lemma 5.3.

6 Bounding the Length of a Clique Path

Let us first recall the various sets we are dealing with. Let (G, k) be an instance of IVD.

- A $(k+2)$-necessary family $\mathcal{W} \subseteq 2^M$ along with a solution M that is 9-redundant with respect to $\mathcal{W}$. In fact, $\mathcal{W} \subseteq 2^M$.
- Every set in $\mathcal{W}$ has size at least 2.
- $\mathcal{C}$ is the set of connected components of $G - M$, $\mathcal{D}$ is the set of connected components in $\mathcal{C}$ that are modules, and $\overline{\mathcal{D}} = \mathcal{C} \setminus \mathcal{D}$. We know that $|V(\mathcal{D})| \leq k^{\mathcal{O}(1)}$ and $|\overline{\mathcal{D}}| \leq k^{\mathcal{O}(1)}$.
- Every maximal clique (and hence every clique) in $G - M$ has size bounded by η.

Let us now turn to the problem of bounding the sizes of non-module components. Observe that to bound this it is sufficient to "bound the length of the clique path'' of a non-module component. This together with the fact that each maximal clique is bounded will lead to the desired result. Our approach mirrors that of [1, 34], but requires additional structural observations corresponding to interval graphs and its obstructions [27, 7]. Each non-module component is a clique path in $G - M$, where M is a 9-redundant modulator.

Let $\mathbb{K} = (K, \beta)$ be a clique path of a non-module component C, where $V(K) = \{x_1, x_2, \ldots, x_t\}$, and for each $i \in [t]$ we let $B_i = \beta(x_i)$. We will refer to the sets B_i, $1 \leq i \leq t$, as the *bags* in $\mathbb{K}$. Any bag B_i in the clique path $\mathbb{K}$ has at most $\eta = 2^{10} \cdot 4(k+5)\binom{|M|}{10}$ vertices (because every maximal clique in $G - M$ has size bounded by η). We let $\beta(\mathbb{K}) = \cup_{i=1}^{t} \beta(x_i)$. Furthermore, for a subpath K' of K, by $\mathbb{K}' = (K', \beta')$ we denote the sub-clique path induced by K'. That is, for $x \in V(K')$, $\beta'(x) = \beta(x)$. Moreover, by $\beta(\mathbb{K}')$ we denote the set $\cup_{x \in V(K')} \beta(x)$. Note that there is a vertex in M that has a neighbor as well as a non-neighbor in C.

In this section, we consider the problem of reducing the number of bags in $\mathbb{K}$. Towards our goal, we will devise a collection of "marking schemes" that mark some polynomially (in k) many bags in $\mathbb{K}$, such that the obstructions are "well behaved" in the region between any two consecutive marked bags. In particular, our marking schemes ensure that if any obstruction intersects an unmarked region of the clique path, then the intersection is an *induced path*. Then, we design reduction rules that "preserve" a minimum separator of the unmarked region. More precisely, we identify an *irrelevant vertex* or an *irrelevant edge*, and then delete it or contract it in the graph. The correctness of these reduction rules follows from the structural properties ensured by the marking schemes.

Let us now define few notations that will be required in this section. Note that these notations apply to $\mathbb{K} = (K, \beta)$ as well as any sub-clique path of it. We fix an ordering (from left to right) of the bags of $\mathbb{K}$, which is given by the path K of the clique path $\mathbb{K}$. We will maintain a set of bags $\mathcal{B}$ in $\mathbb{K}$, which we will call *marked bags*. Initially, $\mathcal{B} = \emptyset$, and we will add some carefully chosen bags in $\mathbb{K}$ to it, as we proceed.

1. For two bags B_ℓ and B_r in $\mathbb{K}$, by $\mathbb{K}[B_\ell, B_r] = (K', \beta')$ we denote the sub-clique path of $\mathbb{K}$ between B_ℓ and B_r (including B_ℓ and B_r).
2. We say that a vertex $v \in \beta(\mathbb{K})$ is a *marked vertex* if there is a marked bag that contains it, otherwise it is an *unmarked vertex*.
3. We say that two marked bags B, B' are *consecutive* if $\mathbb{K}[B, B']$ contains no marked bags other than B and B'.
4. We say that two (distinct) bags B, B in $\mathbb{K}$ are *adjacent* if there is no other bag that lies between them, i.e. $\mathbb{K}[B, B']$ has only two bags, namely, B and B'.
5. For a bag B in $\mathbb{K}$, B^{-1} and B^{+1} denote the bags adjacent to B on its left and right, respectively.

6.1 Partition into Manageable Clique Paths

In this section, we partition the clique path $\mathbb{K}$ into a collection of so called "manageable clique paths", which are well structured with respect to the set M. We will construct a set of marked bags, denoted by $\mathbb{K}(M)$, based on the edges between the vertices in $\beta(\mathbb{K})$ and M. Let us initialize $\mathbb{K}(M)$ as the set containing the first and last bags of $\mathbb{K}$. We begin by stating a property of interval graphs, which will be useful later.

Observation 6.1. *Let H be an interval graph and let H' be the graph obtained by one of the following operations.*

(a) For $v \in V(H)$, $H' = H - \{v\}$.
(b) For $(u, v) \in E(H)$, $H' = H/(u, v)$.

Then H' is an interval graph. Furthermore, the size of any clique in H' is upper-bounded by the size of a maximum clique in H.

The above observation follows from the definition of interval graphs and their interval representation [27]. In particular, statement (b) follows from the observation that an interval representation of $H/(u, v)$ can be obtained by taking an interval representation of H and "merging" the intervals of u and v.

In the following, we will define (auxiliary) graphs that will be helpful in obtaining some useful bags in $\mathbb{K}$. To this end, consider a vertex $m \in M$. Let H_m be the bipartite graph with vertex bipartition $N_G(m) \cap \beta(\mathbb{K})$ and $\beta(\mathbb{K}) \setminus N_G(m)$, where $u \in N_G(m) \cap \beta(\mathbb{K})$ and $v \in \beta(\mathbb{K}) \setminus N_G(m)$ are adjacent in H_m if and only if $(u, v) \in E(G)$. Next, we prove the following lemma about the graph H_m. (Recall that η is an upper bound on the size of any clique in $G - M$.)

Lemma 6.2. *For $m \in M$, let Y_m be a maximum matching in H_m. Then $|Y_m| \leq 2\eta$.*

Proof. Suppose, towards a contradiction, that $|Y_m| > 2\eta$. Let T be the graph obtained from $G[\beta(\mathbb{K})]$ by contracting all the edges in Y_m. Additionally, for each edge (u, v) in Y_m, let w_{uv} be the vertex resulting from its contraction. Recall that $G - M$ is an interval graph of maximum clique size at most η, which together with Observation 6.1 implies that both $G[\beta(\mathbb{K})]$ and T are also interval graphs, and that the maximum size of a clique in these graphs is upper bounded by η. Next, let $\tilde{T}$ be the graph $T[\{w_{uv} \mid (u,v) \in Y_m\}]$. We note that the definition of $\tilde{T}$ relies on the fact that Y_m is a matching in H_m, and thus it has $|Y_m| > 2\eta$ many vertices. From the construction of $\tilde{T}$ and Observation 6.1, it follows that $\tilde{T}$ is also an interval graph and that the size of any clique in $\tilde{T}$ is bounded by η. Interval graphs are perfect graphs, and on a perfect graph $\mathcal{G}$ we know that $\omega(\mathcal{G})\alpha(\mathcal{G}) \geq |V(\mathcal{G})|$, where $\omega(\mathcal{G})$ and $\alpha(\mathcal{G})$ denote the size of a maximum clique and a maximum independent set in $\mathcal{G}$, respectively [45] (or Theorem 3.3 [27]). This implies that there is an independent set in $\tilde{T}$ of size at least $|Y_m|/\eta > 2$. Consider an independent set of size 3 in $\tilde{T}$, and the corresponding edges of the matching Y_m. It follows that these three edges and the vertex m form a long claw, $\mathbb{O}$ in G, which is an obstruction of size 7. Since Reduction Rule 3.1 is not applicable, each set in $\mathcal{W}$ is of size at least 2. Moreover, $|V(\mathbb{O}) \cap M| = 1$. Therefore, $\mathbb{O}$ is not covered by $\mathcal{W}$. But then, since M is a 9-redundant solution each obstruction in G which is not covered by $\mathcal{W}$ must contain at least 10 vertices from M. Thus, we deduce that $|Y_m| > 2\eta$ cannot hold. □

For each $m \in M$, we compute a maximum matching Y_m in the graph H_m. Then for each edge in Y_m we pick a bag in $\mathbb{K}$ that contains this edge and add it to $\mathbb{K}(M)$. Let us observe that we have added at most $2\eta|M|$ bags to $\mathbb{K}(M)$. Before proceeding further, we add some more bags to $\mathbb{K}(M)$ that give us some additional structural properties. Next, we state the following observation, which will be useful in designing one of our marking schemes for bags in $\mathbb{K}$.

Observation 6.3. *Let $m_1, m_2 \in M$ be (distinct) vertices such that $\{m_1, m_2\} \notin \mathcal{W}$ and $(m_1, m_2) \notin E(G)$. Then, $(N_G(m_1) \cap N_G(m_2)) \setminus M$ induces a clique in G.*

Proof. This observation is the special case of Lemma 4.6 with $M' = M, u = m_1, v = m_2$ and $u, v \in M$. □

Next, consider (distinct) $m_1, m_2 \in M$, such that $\{m_1, m_2\} \notin \mathcal{W}$ and $(m_1, m_2) \notin E(G)$. Let $B(m_1, m_2)$ be a bag in $\mathbb{K}$, such that $(N_G(m_1) \cap N_G(m_2)) \cap \beta(\mathbb{K}) \subseteq B(m_1, m_2)$. We note that the existence of $B(m_1, m_2)$ is guaranteed from Observation 6.3. We add $B(m_1, m_2)$ to the set $\mathbb{K}(M)$. We are now ready to state our first bag-marking scheme.

Marking Scheme I. Add all the bags in $\mathbb{K}(M)$ to $\mathcal{B}$.

Note that $|\mathbb{K}(M)|$ is at most $2\eta|M| + |M|^2 + 2$. This bound is obtained because *(i)* $\mathbb{K}(M)$ contains the first and last bag of $\mathbb{K}$, *(ii)* at most 2η bags in $\mathbb{K}$ were added corresponding to the matching Y_m for each $m \in M$ (and H_m), and *(iii)* for (distinct) $m_1, m_2 \in M$, such that $\{m_1, m_2\} \notin \mathcal{W}$ and $(m_1, m_2) \notin E(G)$, we added a bag to $\mathbb{K}(M)$. Thus, using Marking Scheme I, we have marked at most $\boxed{2\eta|M| + |M|^2 + 2 < 4\eta|M|}$ bags in $\mathbb{K}$. Here, we used the fact that $\eta \geq |M|$.

Next, we state an observation regarding vertices which are not present in any bag in $\mathbb{K}(M)$, which will be useful later. We note that this observation is very similar to Observation 4.10 of Section 4.

Observation 6.4. *Consider a vertex $v \in \beta(\mathbb{K})$ such that there is no bag in $\mathbb{K}(M)$ that contains v. For (distinct) vertices $u, w \in N_G(v) \cap M$, at least one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds.*

Proof. Consider $v \in \beta(\mathbb{K})$ such that there is no bag in $\mathbb{K}(M)$ that contains v, and (distinct) vertices $u, w \in N_G(v) \cap M$. Suppose, by way of contradiction, that $\{u, w\} \notin \mathcal{W}$ and $(u, v) \notin E(G)$. This together with Observation 6.3, implies that $(N_G(u) \cap N_G(w)) \setminus M$ induces a clique in G. From the above and Marking Scheme I, it follows that there is a bag $B(u, w)$ in $\mathbb{K}(M)$ such that $(N_G(u) \cap N_G(w)) \setminus M \subseteq B(u, w)$. However, $v \in (N_G(u) \cap N_G(w)) \setminus M$ and hence $v \in B(u, w)$. This contradicts that v is not contained in any bag in $\mathbb{K}(M)$. □

Let $B_\ell, B_r \in \mathbb{K}(M)$ be two consecutive marked bags in $\mathbb{K}$. We define the graph $G[B_\ell, B_r]$ to be the graph induced on the vertices appearing in the sub-clique path $\mathbb{K}[B_\ell, B_r]$ excluding the vertices in B_ℓ and B_r. That is, $G[B_\ell, B_r] = G[V[B_\ell, B_r]]$ where $V[B_\ell, B_r] = \beta(\mathbb{K}[B_\ell, B_r]) \setminus (B_\ell \cup B_r)$. Note that although $G[\beta(\mathbb{K}[B_\ell, B_r])]$ is a connected subgraph of G, $G[B_\ell, B_r]$ need not be connected graph. We refer to a connected component of $G[B_\ell, B_r]$ as an *obstruded component of* $\mathbb{K}[B_\ell, B_r]$. We extend this definition to say that an induced subgraph H of $G[\beta(\mathbb{K})]$ is an *obstruded component of* $\mathbb{K}$, if there are consecutive marked bags $B_\ell, B_r \in \mathbb{K}(M)$, such that H is an obstruded component of $\mathbb{K}[B_\ell, B_r]$. In the following, we prove a property regarding the obstruded components of $\mathbb{K}$.

Lemma 6.5. *Let H be an obstruded component of $\mathbb{K}$. For each $m \in M$, either we have $V(H) \subseteq N_G(m)$ or we have $V(H) \cap N_G(m) = \emptyset$.*

Proof. Suppose, towards a contradiction, that there exists $m \in M$ that has both a neighbor and a non-neighbor from the set $V(H)$ in G. Because H is connected, this implies that there is an edge $e \in E(H)$ such that one endpoint of e lies in $N_G(m) \cap \beta(\mathbb{K})$ and the other endpoint of e lies in $\beta(\mathbb{K}) \setminus N_G(m)$, i.e. $e \in E(H_m)$. Furthermore, by construction, both these endpoints are different from all the vertices belonging to the edges of the matching Y_m in H_m. Therefore $Y_m \cup \{e\}$ is also a matching in H_m. However, this is a contradiction as Y_m is a maximum matching in H_m. This concludes the proof. $\square$

Let us fix a pair of consecutive marked bags $B_\ell, B_r \in \mathbb{K}(M)$ and consider the obstruded components of $\mathbb{K}[B_\ell, B_r]$. Note that Lemma 6.5 can be interpreted as follows. Any obstruded component of $\mathbb{K}[B_\ell, B_r]$ is a "module with respect to M". The following lemma shows that all but at most 4η of these obstruded components are actually modules in the graph G.

Lemma 6.6. *All but at most 4η of the obstruded components of $\mathbb{K}[B_\ell, B_r]$ are modules in G.*

Proof. Let H be an obstruded component of $\mathbb{K}[B_\ell, B_r]$. For any vertex $v \in B_\ell \cup B_r$ there are at most two obstruded components in $\mathbb{K}[B_\ell, B_r]$ with the property that v has both a neighbor and a non-neighbor in the component. Indeed, if this were not the case, then we would have obtained a long claw in $G[\beta(\mathbb{K})] - M$, which is a contradiction. Notice that there are at most 2η vertices in $B_\ell \cup B_r$. Hence, it follows that all but at most 4η obstruded components of $\mathbb{K}[B_\ell, B_r]$ have the following property: Each vertex $v \in B_\ell \cup B_r$ is adjacent either to all vertices of this obstruded component or to none of them. Finally, observe that the neighborhood of a vertex in an obstruded component H, excluding the neighbors that belong to H itself, is a subset of $M \cup B_\ell \cup B_r$. Hence, it follows from the above arguments and Lemma 6.5, that all but at most 4η obstruded components of $\mathbb{K}[B_\ell, B_r]$ are modules in G. $\square$

Let us note another useful property of the obstruded components.

Lemma 6.7. *Let H be an obstruded component of $\mathbb{K}[B_\ell, B_r]$. Then there is a sub-clique path $\mathbb{K}_H$ of $\mathbb{K}[B_\ell, B_r]$ such that $V(H) \subseteq \beta(\mathbb{K}_H) \subseteq V(H) \cup B_\ell \cup B_r$.*

Proof. Since H is a connected graph and $\mathbb{K}$ is a path-decomposition, it follows from the definition of a path-decomposition that the set of bags of $\mathbb{K}$ that have nonempty intersection with $V(H)$ forms a sub-clique path $\mathbb{K}_H$ of $\mathbb{K}$. Furthermore, as H is a connected component of $G[B_\ell, B_r] = G[V[B_\ell, B_r]]$ where $V[B_\ell, B_r] = \beta(\mathbb{K}[B_\ell, B_r]) \setminus (B_\ell \cup B_r)$, it follows that $V(H) = \beta(\mathbb{K}_H) \setminus (B_\ell \cup B_r)$. Therefore, $\mathbb{K}_H$ is a sub-clique path of $\mathbb{K}[B_\ell, B_r]$ and $V(H) \subseteq \beta(\mathbb{K}_H) \subseteq V(H) \cup B_\ell \cup B_r$. $\square$

The obstruded components of $\mathbb{K}[B_\ell, B_r]$ can be divided into two groups, those that are modules in G and the rest. We will first consider the problem of reducing the module obstruded components.

6.1.1 Handling Obstruded Modules of $\mathbb{K}$

In this subsection, our goal will be to upper bound the total vertices across all bags B that have that following property: B has nonempty intersection with at least one obstruded component of $\mathbb{K}$ that is a module in G. First, we will only reduce the total number of vertices in the obstruded components of $\mathbb{K}$ that are modules in G. To achieve this, we will employ Lemma 4.3 (see Section 4). To this end, consider a pair of consecutive marked bags B_ℓ, B_r in $\mathbb{K}(M)$. Let $\widehat{\mathcal{C}}$ be the set of obstruded components of $\mathbb{K}[B_\ell, B_r]$ that are modules in G. Note that by the construction, $\widehat{\mathcal{C}}$ is the set of connected components in $G[B_\ell, B_r] = G[V[B_\ell, B_r]]$ (where $V[B_\ell, B_r] = \beta(\mathbb{K}[B_\ell, B_r]) \setminus (B_\ell \cup B_r)$) that are modules. Thus, from the definition of a path decomposition, it follows that $\widehat{\mathcal{C}}$ is a set of connected components in $G - (M \cup B_\ell \cup B_r)$ that are modules. Moreover, note that $|M \cup B_\ell \cup B_r| \leq |M| + 2\eta$.

Now we apply Lemma 4.3 for $\widehat{M} = B_\ell \cup B_r$, and obtain a subset Z of $V(\widehat{\mathcal{C}})$ of size at most $4(k+2)^2(|M|+2\eta)^6$ such that the following holds:

> If $S \subseteq V(G)$ of size at most k and $\mathbb{O}$ is an obstruction in $G - S$ that is not covered by $\mathcal{W}$, then there is another obstruction $\mathbb{O}'$ in $G - S$ such that $\mathbb{O}' \cap (V(\widehat{\mathcal{C}}) \setminus Z) = \emptyset$.

This gives the following reduction rule.

Reduction Rule 6.1. *Suppose there is $v \in V(\widehat{\mathcal{C}}) \setminus Z$. Then, delete v from the graph G. That is, the resulting instance is $(G - \{v\}, k)$.*

Lemma 6.8. *Reduction rule 6.1 is safe.*

Proof. Let $v \in V(\widehat{\mathcal{C}}) \setminus Z$, and $G' = G - \{v\}$. We will show that (G, k) is a Yes-instance of IVD if and only if (G', k) is. In the forward direction, let S be a solution to (G, k). As $G' - S$ is an induced subgraph of $G - S$, Observation 6.1 implies that S is a solution to (G', k).

In the reverse direction, let S' be a solution to (G', k). We claim that S' is a solution to (G, k). Towards a contradiction, suppose that this claim is false. Then, there is an obstruction $\mathbb{O}$ in $G - S'$. Notice that $\mathbb{O}$ is not covered by $\mathcal{W}$—indeed, if $\mathbb{O}$ were covered by $\mathcal{W}$, then because $S \cap M = S' \cap M$ and $\mathcal{W} \subseteq 2^M$ is a $(k+2)$-necessary family, it would have followed that $V(\mathbb{O}) \cap S' \neq \emptyset$. Thus, Lemma 4.3 implies that there is an obstruction $\mathbb{O}'$ in $G - S'$ that is disjoint from $V(\widehat{\mathcal{C}}) \setminus Z$. The obstruction $\mathbb{O}'$ does not contain the vertex v, hence it is also an obstruction in $(G - \{v\}) - S' = G - S$. Since we have reached a contradiction, the proof i complete. $\square$

If Reduction Rule 6.1 is not applicable, then we can assume that the (total) number of vertices in $V(\widehat{\mathcal{C}})$ is bounded by $4(k+1)^2(|M|+2\eta)^6$. In the following lemma, we bound the number of bags in $\mathbb{K}$ that have nonempty intersection with $V(\widehat{\mathcal{C}})$.

Lemma 6.9. *The number of bags in $\mathbb{K}$ having nonempty intersection with $V(\widehat{\mathcal{C}})$ is bounded by $12|V(\widehat{\mathcal{C}})|$.*

Proof. Let us first note that any bag in $\mathbb{K}$ that contains at least one vertex of $V(\widehat{\mathcal{C}})$ is a subset of $V(\widehat{\mathcal{C}}) \cup B_\ell \cup B_r$ and is also a bag in $\mathbb{K}[B_\ell, B_r]$. To prove the desired claim, we create a special set of bags $\mathcal{S}$, as follows. Firstly, add B_ℓ, B_r to $\mathcal{S}$. Without loss of generality we assume that B_ℓ appears before B_r in the ordering of the bags given by $\mathbb{K}$. For each $x \in B_\ell$, let B_x be the first bag in $\mathbb{K}[B_\ell, B_r]$ which does *not* contain x, where if such a bag does not exist, then we set $B_x = B_r$. Similarly, for each $y \in B_r$, let $\widehat{B}_y$ be the first bag in $\mathbb{K}[B_\ell, B_r]$ which contains y, which exists since $y \in B_r$. We add all the bags in $\{B_x \mid x \in B_\ell\} \cup \{\widehat{B}_y \mid y \in B_r\}$ to $\mathcal{S}$. Next, for each $v \in V(\widehat{\mathcal{C}})$ let F_v and L_v be the first bag and last bag in $\mathbb{K}[B_\ell, B_r]$ containing v, respectively. We further add each bag in $\{F_v \mid v \in V(\widehat{\mathcal{C}})\} \cup \{L_v \mid v \in V(\widehat{\mathcal{C}})\}$ to $\mathcal{S}$. Notice that $|\mathcal{S}| \leq |B_\ell| + |B_r| + 2|V(\widehat{\mathcal{C}})| + 2 \leq 6|V(\widehat{\mathcal{C}})|$. Consider any two bags B_1, B_2 in

$\mathcal{S}$ (where B_1 appears before B_2 in the ordering given by $\mathbb{K}$) such that there is no bag from $\mathcal{S}$ in $\mathbb{K}[B_1, B_2]$ other than B_1 and B_2. We call the sub-clique path $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ (which might be empty) a *restricted region* of the sub-clique path $\mathbb{K}[B_\ell, B_r]$. In the following, we will argue that all the bags belonging to the same restricted region contain the same set of vertices from $B_\ell \cup B_r \cup V(\widehat{\mathcal{C}})$. We make this argument with respect to B_1 and B_2. To this end, consider the set $\mathcal{S}' = \{X \in \mathbb{K}[B_1, B_2] \mid X \notin \{B_1, B_2\}\}$. We will argue that for any $X, Y \in \mathcal{S}'$, we have $X \cap (B_\ell \cup B_r \cup V(\widehat{\mathcal{C}})) = Y \cap (B_\ell \cup B_r \cup V(\widehat{\mathcal{C}}))$. Towards this, consider some $X, Y \in \mathcal{S}'$ such that X appears before Y in the ordering given by $\mathbb{K}$. We consider two cases as follows, and in each of the cases we rely on the property that in a clique path, the set of bags containing a fixed vertex forms a sub-clique path.

- There is $v \in (X \setminus Y) \cap (B_\ell \cup B_r \cup V(\widehat{\mathcal{C}}))$. Note that $v \notin B_r$ as otherwise it belongs to $X \cap B_r$ but not to Y, which violates the sub-clique path property of a clique path. Consider the subcase where $v \in B_\ell$. This implies that v belongs to each bag in $\mathbb{K}[B_\ell, X]$. But as $v \notin Y$, the bag $B_v \in \mathcal{S}$ must belong to $\mathbb{K}[X, Y]$. This contradicts the fact that $\mathbb{K}[X, Y]$ does not contain any bag from $\mathcal{S}$. Next, consider the subcase where $v \in V(\widehat{\mathcal{C}})$. Again, as $v \in X$ and $v \notin Y$, we have that the bag L_v must belong to $\mathbb{K}[X, Y]$, which is a contradiction.

- There is $v \in (Y \setminus X) \cap (B_\ell \cup B_r \cup V(\widehat{\mathcal{C}}))$. Note that $v \notin B_\ell$ as otherwise it belongs to $B_\ell \cap Y$ but not to X, which violates the sub-clique path property of a clique path. Consider the subcase where $v \in B_r$. This implies that v belongs to each bag in $\mathbb{K}[Y, B_r]$. But as $v \notin X$, the bag $\widehat{B}_v \in \mathcal{S}$ must belong to $\mathbb{K}[X, Y]$. This contradicts the fact that $\mathbb{K}[X, Y]$ does not contain any bag from $\mathcal{S}$. Next, consider the subcase where $v \in V(\widehat{\mathcal{C}})$. Again, as $v \notin X$ and $v \in Y$, we have that the bag F_v must belong to $\mathbb{K}[X, Y]$, which is a contradiction.

From the above we conclude that bags in the same restricted region contain the same set of vertices from $B_\ell \cup B_r \cup V(\widehat{\mathcal{C}})$. In what follows, we will show why this statement implies that in any restricted region, there can be at most one bag that has nonempty intersection with $V(\widehat{\mathcal{C}})$. Before showing that this claim is true, let us argue that having this claim concludes the proof. Indeed, since $|\mathcal{S}| \leq 6|V(\widehat{\mathcal{C}})|$ and $B_\ell, B_r \in \mathcal{S}$, there are at most $6|V(\widehat{\mathcal{C}})|$ restricted regions that can have nonempty intersection with $V(\widehat{\mathcal{C}})$. Each one of these regions has only one bag that has nonempty intersection with $V(\widehat{\mathcal{C}})$. Adding up the bags in $\mathcal{S}$ itself, we conclude that there are at most $12|V(\widehat{\mathcal{C}})|$ bags in $\mathbb{K}$ that contain a vertex from $V(\widehat{\mathcal{C}})$.

We now turn to show that in any restricted region, there can be at most one bag that has nonempty intersection with $V(\widehat{\mathcal{C}})$. For this purpose, consider some restricted region $\mathbb{K}[B_1^{+1}, B_2^{-1}]$. Then, all bags in this region contain the same set of vertices from $B_\ell \cup B_r \cup V(\widehat{\mathcal{C}})$. Suppose that this region contains some vertex $v \in V(\widehat{\mathcal{C}})$. By the definition of $\widehat{\mathcal{C}}$, there exists an obstruded component H of $\mathbb{K}[B_\ell, B_r]$ that contains v. Because v belongs to every bag in $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ and by Lemma 6.7, it follows that H contains all vertices across all bags in $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ apart from those in $B_\ell \cup B_r$. Thus, all vertices across all bags in $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ belong to $B_\ell \cup B_r \cup V(\widehat{\mathcal{C}})$. Because distinct bags on a clique path correspond to distinct sets of vertices, this means that $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ can only contain a single bag. This concludes the proof. □

Recall that there are at most $4\eta|M|$ pairs of consecutive marked bags in $\mathbb{K}(M)$. Applying Reduction Rule 6.1 for every such pair, we obtain the following. There are at most $4(k+1)^2(|M|+2\eta)^6 \cdot 4\eta|M|$ vertices in $\mathbb{K}$ that lie in the union of all module components. Let $\mathcal{C}(\mathbb{K})$ denote this collection of vertices.

Marking Scheme II. Add all the bags in $\mathbb{K}$ that contain a vertex from $\mathcal{C}(\mathbb{K})$ to $\mathcal{B}$.

From Lemma 6.9, we obtain that we have marked at most $\boxed{48(k+1)^2(|M|+2\eta)^6 \cdot 4\eta|M|}$ bags of $\mathbb{K}$, using Marking Scheme II.

6.1.2 Obtaining Manageable Clique Paths

In this subsection, we will focus on the obstruded components of $\mathbb{K}$ that are not modules in G. To this end, we mark some more bags in $\mathbb{K}$ so that the regions between unmarked bags have additional structural properties. We will refer to the sub-clique paths obtained by this process as *manageable clique paths*. In the following, we start by defining some notation which will be helpful in describing this marking scheme.

Let B_ℓ, B_r be two consecutive marked bags in $\mathbb{K}(M)$, where B_ℓ appears before B_r in the ordering given by $\mathbb{K}$. Next, consider a non-module obstruded component X of $\mathbb{K}[B_\ell, B_r]$ (of course it contains an unmarked vertex), and let $\mathbb{K}_X$ be the sub-clique path of $\mathbb{K}[B_\ell, B_r]$ provided by Lemma 6.7. Let B_ℓ^X and B_r^X be the first and last bags of $\mathbb{K}_X$, respectively. Before moving on to our next marking scheme, we construct two sets of bags, $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$. Initially, we have $\mathcal{L}_1(X) = \{B_\ell^X, B_r^X\}$. We note that the construction of $\mathcal{L}_1(X)$ is very similar to the construction of $\mathcal{S}$ used in the proof of Lemma 6.9. For each $u \in B_\ell$, let $B_u(X)$ be the first bag in $\mathbb{K}_X$ which does not contain u, where if such a bag does not exist, then we set $B_u(X) = B_r^X$. Additionally, for each $v \in B_r \setminus B_\ell$, let $B_v(X)$ to be the first bag in $\mathbb{K}_X$ which contains v, where if such a bag does not exists, then we set $B_v = B_r^X$. We add all the bags in $\{B_u(X) \mid u \in B_\ell\} \cup \{B_v(X) \mid v \in B_r \setminus B_\ell\}$ to $\mathcal{L}_1(X)$. We initialize $\mathcal{L}_2(X) = \mathcal{L}_1(X)$. Furthermore, for each bag $B \in \mathcal{L}_1(X)$ in $\mathbb{K}_X$, we add to $\mathcal{L}_2(X)$ the bags adjacent to B, namely B^{-1} and B^{+1} (if they exist) in $\mathbb{K}_X$. Note that the number of bags in $\mathcal{L}_2(X)$ is bounded by 10η.

For consecutive unmarked bags B_ℓ, B_r in $\mathbb{K}$ (considering only marked bags in $\mathbb{K}(M)$) let $\mathcal{X}(B_\ell, B_r)$ be the set of non-module obstruded components of $\mathbb{K}[B_\ell, B_r]$. Furthermore, let $\mathcal{L}(B_\ell, B_r)$ be the union of the sets $\mathcal{L}_2(X)$ taken over all $X \in \mathcal{X}(B_\ell, B_r)$. From Lemma 6.6, we know that there are at most 4η obstruded components of $\mathbb{K}[B_\ell, B_r]$ that are not modules. Thus, the number of bags in $\mathcal{L}(B_\ell, B_r)$ is bounded by $40\eta^2$. Finally, let $\widehat{\mathbb{K}(M)}$ be the union of the sets of bags $\mathcal{L}(B_\ell, B_r)$ taken over all B_ℓ and B_r that are consecutive unmarked bags in $\mathbb{K}$ (considering only marked bags in $\mathbb{K}(M)$). Recall that $|\mathbb{K}(M)|$ is bounded by $4\eta|M|$. Thus, the number of bags in $\widehat{\mathbb{K}(M)}$ is bounded by $160\eta^3|M|$. We are now ready to state our third marking scheme.

Marking Scheme III. Add all the bags in $\widehat{\mathbb{K}(M)}$ to $\mathcal{B}$.

Note that we marked at most $\boxed{160\eta^3|M|}$ bags using the above marking scheme. We now further partition $\mathbb{K}$ using the bags marked in the above scheme.

In the following we will give some useful properties regarding the region between marked consecutive bags. To this end, let B_ℓ, B_r be consecutive marked bags in $\mathbb{K}$, where we consider marked bags only in $\mathbb{K}(M)$. We assume that B_ℓ appears before B_r in the ordering given by $\mathbb{K}$. Consider an obstruded non-module component of $\mathbb{K}[B_\ell, B_r]$, and let $\mathbb{K}_X$ be the sub-clique path provided by Lemma 6.7. Furthermore, consider two consecutive marked bags $B_p, B_q \in \widehat{\mathbb{K}(M)}$ in $\mathbb{K}_X$. The sub-clique path $\mathbb{K}_X[B_p, B_q]$ is called a (B_ℓ, B_r)-*manageable clique path* (or simply, a *manageable clique path*) if $\mathbb{K}_X[B_p, B_q]$ contains at least one bag apart from B_p and B_q. Next, we derive the following property using the notations we introduced above.

Lemma 6.10. *For any two bags B, B' in a (B_ℓ, B_r)-manageable clique path $\mathbb{K}_X[B_p, B_q]$, we have $B \cap (B_\ell \cup B_r) = B' \cap (B_\ell \cup B_r)$.*

Proof. Note that since B_p appears before B_q, and $\mathbb{K}_X[B_p, B_q]$ contains at least three bags, we have that $B_p^{-1}, B_q^{+1} \in \mathcal{L}_1(X) \subseteq \widehat{\mathbb{K}(M)}$. From this we derive that $\mathbb{K}_X[B_p, B_q]$ contains no bag from $\mathcal{L}_1(X)$. Consider two bags S, T in $\mathbb{K}_X[B_p, B_q]$ such that S appears before T in the ordering given by $\mathbb{K}$. We consider the following cases, and in each of the cases we rely on the property that in a clique path, the set of bags containing a fixed vertex forms a sub-clique path.

- There is $v \in (S \setminus T) \cap (B_\ell \cup B_r)$. Note that $v \notin B_r$ as otherwise it belongs to $S \cap B_r$ but not to T, which violates the sub-clique path property of a clique path. Consider the case when $v \in B_\ell$. This implies that v belongs to each bag in $\mathbb{K}[B_\ell, S]$. But as $v \notin T$, the bag $B_v(X) \in \mathcal{L}_1(X)$ must belong to $\mathbb{K}[S,T]$. This contradicts the fact that $\mathbb{K}_X[B_p, B_q]$ does not contain any bag from $\mathcal{L}_1(X)$.

- There is $v \in (T \setminus S) \cap (B_\ell \cup B_r)$. Note that $v \notin B_\ell$ as otherwise it belongs to $T \cap B_\ell$ but not to S, which violates the sub-clique path property of a clique path. Consider the case when $v \in B_r$. This implies that v belongs to each bag in $\mathbb{K}[T, B_r]$. But as $v \notin S$, the bag $B_v(X) \in \mathcal{L}_1(X)$ must belong to $\mathbb{K}[S,T]$. This contradicts the fact that $\mathbb{K}_X[B_p, B_q]$ does not contain any bag from $\mathcal{L}_1(X)$.

This concludes the proof. $\square$

We will conclude this subsection by deriving few more properties of manageable clique paths, which will be useful later. Consider a (B_ℓ, B_r)-manageable clique path $\mathbb{K}_X$, and let $C_X = B_\ell \cap B_r$. (Here, 'C' stands for "common".) **Note that here, $\mathbb{K}_X$ is not the clique path provided by Lemma 6.7 but a subclique path of the clique path provided by that lemma as per the definition of a (B_ℓ, B_r)-manageable clique path.**[3] We have the following observation, that follows from the construction of $\mathbb{K}_X$ (together with the Marking Scheme I), Observation 6.4, and Lemma 6.5.

Observation 6.11. *For $m \in M$, either $\beta(\mathbb{K}_X) \setminus C_X \subseteq N_G(m)$ or $(\beta(\mathbb{K}_X) \setminus C_X) \cap N_G(m) = \emptyset$. Furthermore, for $v \in \beta(\mathbb{K}_X) \setminus C_X$ and $u, w \in N_G(v) \cap M$, at least one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds.*

Proof. The first part of the observation follows from Lemma 6.5, and the second part of the observation follows from Observation 6.4. $\square$

Let us define $M_A = M \cap N(\beta(\mathbb{K}_X) \setminus C_X)$, and $M_P = M \setminus M_A$. (Here, 'A' stands for "all" and 'P' stands for "private".) Let us observe that, by construction, $N(M_P) \cap \beta(\mathbb{K}_X) \subseteq C_X$. We note that there may be a vertex $v \in C_X$ and a vertex $m \in M_A$ such that $(v, m) \notin E(G)$.

Observation 6.12. *Consider $v \in C_X$ and $m \in M_A$ such that $(v, m) \notin E(G)$. Then, $G[\beta(\mathbb{K}_X)]$ is a clique in G.*

Proof. Notice that $C_X \subseteq B_\ell$ and B_ℓ is a clique in G, and thus $G[C_X]$ is a clique. Also, every vertex in $C_X \cup M_A$ is adjacent to every vertex in $\beta(\mathbb{K}_X) \setminus C_X$ in the graph G. Therefore, if there is a pair of non-adjacent vertices $u, w \in \beta(\mathbb{K}_X) \setminus C_X$, then $\mathbb{O} = G[\{u, v, w, m\}]$ is an induced cycle on four vertices. Since Reduction Rule 3.1 is not applicable, each set in $\mathcal{W}$ has size at least 2, and hence $\mathbb{O}$ is not covered by $\mathcal{W}$. But then any obstruction which is not covered by $\mathcal{W}$ must intersect M in at least ten vertices. Hence, we arrive at a contradiction. $\square$

Next, we summarize some properties regarding those manageable clique paths $\mathbb{K}_X$ for which $G[\beta(\mathbb{K}_X)]$ is not a clique.

Observation 6.13. *Suppose $G[\beta(\mathbb{K}_X)]$ is not a clique. Then each of the following holds.*

1. *For any $v \in \beta(\mathbb{K}_X)$ and $m \in M_A$, we have $(v, m) \in E(G)$.*
2. *For each $u \in C_X$ and $v \in \beta(\mathbb{K}_X)$, we have $(u, v) \in E(G)$.*
3. *For each $m_1, m_2 \in M_A$, at least one of $\{m_1, m_2\} \in \mathcal{W}$ or $(m_1, m_2) \in E(G)$ holds.*

Proof. The first item follows from Observation 6.12 because $G[\beta(\mathbb{K}_X)]$ is not a clique. Since C_X is a clique that is contained in every bag of $\mathbb{K}_X$ in G, the second item of the observation follows. Lastly, the third item follow from Observation 6.11 and the definition of M_A. $\square$

[3] We use this abbreviation rather than the notation $\mathbb{K}_X[B_p, B_q]$ in Lemma 6.10 in order to lighten notation.

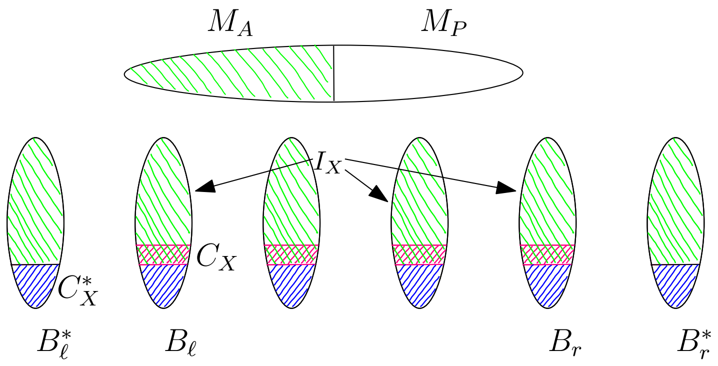

Figure 12: A component $\mathbb{K}[B_\ell^*, B_r^*]$ with a (B_ℓ^*, B_r^*)-manageable clique path $\mathbb{K}_X = \mathbb{K}[B_\ell, B_r]$.

6.2 Handling Manageable Clique Paths

We start by recalling that the number of manageable clique paths is bounded by $160\eta^3|M|$. Notice that a manageable clique path $\mathbb{K}_X$ that induces a clique in $G - M$ must consist of only a single bag in $\mathbb{K}$. We let $\widetilde{\mathbb{K}(M)}$ be the set of bags in $\mathbb{K}$ comprising of every manageable clique path that induces a clique in $G - M$. We state our next marking scheme.

Marking Scheme IV. Add all the bags in $\widetilde{\mathbb{K}(M)}$ to $\mathcal{B}$.

We note that by the above marking scheme we have marked at most $\boxed{160\eta^3|M|}$ many bags. Hereafter, we deal with only those manageable clique paths that do not induce a clique in G.

In the following, consider a manageable clique path $\mathbb{K}_X = \mathbb{K}[B_\ell, B_r]$ that does not induce a clique, whose first and last bags are B_ℓ and B_r, respectively. **Note that here, B_ℓ and B_r are the first and last bags of the manageable clique path, respectively, and not necessarily the consecutive bags of $\mathbb{K}(M)$ as before that define the larger region in which the manageable clique path lies. We make this change to ensure that B_ℓ and B_r always denote the end bags of the sub-clique path that is our current focus.** With the exception of the next paragraph (where we provide appropriate clarifying explanations), this will not cause confusion later.

Let $C_X = B_\ell \cap B_r$ and $I_X = \beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$. (Here, '$C$' stands for "common" and 'I' stands for "internal".) Observe that no vertex in I_X belongs to any marked bag (among all bags marked so far). Let $M_A = M \cap N(\beta(\mathbb{K}_X) \setminus C_X)$, and $M_P = M \setminus M_A$. Let us argue that no confusion shall arise when dealing with this M_A and the one in Observation 6.11, since they are actually equal. To this end, let B_ℓ^* and B_r^* be the consecutive marked bags in $\mathbb{K}(M)$ with respect to whom $\mathbb{K}_X$ is a (B_ℓ^*, B_R^*)-manageable path. Accordingly, denote $C_X^* = B_\ell^* \cap B_r^*$ and $M_A^* = M \cap N(\beta(\mathbb{K}_X) \setminus C_X^*)$ (see Fig. 12). Then, we have the following observation.

Observation 6.14. *With respect to the notation above, we have that $C_X^* \subseteq C_X$ and $M_A = M_A^*$.*

Proof. By the definition of a path decomposition, any vertex that belongs to both B_ℓ^* and B_r^* must also belong to every bag in between these two bags, and in particular to both B_ℓ and B_r. Thus, it follows that $C_X^* \subseteq C_X$. This containment directly implies that $M_A \subseteq M_A^*$. However, we need to show that M_A and M_A^* are in fact equal. To this end, consider a vertex $m \in M_A^*$. By Observation 6.11, we have that $\beta(\mathbb{K}_X) \setminus C_X^* \subseteq N_G(m)$, and therefore $\beta(\mathbb{K}_X) \setminus C_X \subseteq N_G(m)$. Thus, unless $\beta(\mathbb{K}_X) \setminus C_X$ is empty, the last containment implies that $m \in M_A$. However, $\beta(\mathbb{K}_X) \setminus C_X$ cannot be empty, since then $\mathbb{K}_X$ would have induced a clique. $\square$

In light of this observation, from Observation 6.11 and 6.13 it follows that M_A is adjacent to every vertex in $\beta(\mathbb{K}_X)$ and $N(M_P) \cap \beta(\mathbb{K}_X) \subseteq C_X$. Furthermore, from Observation 6.11, for each $m_1, m_2 \in M_A$, one of $\{m_1, m_2\} \in \mathcal{W}$ or $(m_1, m_2) \in E(G)$ holds.

We will devise a sequence of marking schemes that mark a polynomial in k number of bags in $\mathbb{K}_X$, such that the obstructions are "well-behaved" with respect to the marked bags. Towards that, we have the following definition for an AW.

Definition 6.1. For a manageable clique path $\mathbb{K}_X$, an obstruction $\mathbb{O}$ is called $\mathbb{K}_X$*-manageable* if all terminals in $\mathbb{O} \cap \beta(\mathbb{K}_X)$ appear in the marked bags. Furthermore, it is called a *manageable obstruction* if it is $\mathbb{K}_X$-manageable for every manageable clique path $\mathbb{K}_X$.

More precisely, our goal is to mark a polynomial in k number of bags in $\mathbb{K}_X$ so that, for any set S of $k+2$ or fewer vertices, if there is an AW $\mathbb{O}$ in $G - S$, then there must be another AW $\mathbb{O}$ in $G - S$ that is $\mathbb{K}_X$-manageable. Let $\mathcal{B}_X$ be the set of marked bags in $\mathbb{K}_X$, and note that initially $\mathcal{B}_X$ contains only the first bag B_ℓ of $\mathbb{K}_X$ and last bag B_r of $\mathbb{K}_X$.

We present the following Lemma 6.15 to characterize the intersection between a manageable clique path $\mathbb{K}_X$ and an induced path P in G. Let us note the this lemma holds for any sub-clique path $\mathbb{K}_{X'}$ of $\mathbb{K}_X$ where we define the associated vertex sets $B'_\ell, B'_r, I_{X'}$ and $C_{X'}$ accordingly; since the arguments remain the same, we only consider $\mathbb{K}_X$ in the proof. Note that the sets M_A and M_P remain unchanged for $\mathbb{K}_{X'}$ (this follows from the arguments given in the proof of Observation 6.14). **We remark that here, we slightly abuse notation—$\mathbb{K}_{X'}$ refers to a sub-clique path of $\mathbb{K}_X$ rather than a sub-clique path of $\mathbb{K}$ associated with a new obstruded component $X' \neq X$. This is done so that we can conveniently use the notations $I_{X'}$ and $C_{X'}$ in the same way as before. Since we no longer refer explicitly to obstruded components, but have zoomed into a manageable clique-path in the context of one such component, confusion is avoided.** Here, we deal with induced paths that have at least one internal vertex that is also an interval vertex in $\mathbb{K}_X$ but which do not consist only of internal vertices in $\mathbb{K}_X$; moreover, we forbid the paths from traversing M_A.

Lemma 6.15. *Let $P = (v_1, v_2, \ldots, v_t)$ be an induced path in G, such that all the following conditions are satisfied:*

1. *$(V(P) \setminus \{v_1, v_t\}) \cap I_X \neq \emptyset$,*
2. *$V(P) \cap M_A = \emptyset$, and*
3. *$V(P) \cap (V(G) \setminus I_X) \neq \emptyset$.*

Then, $V(P) \cap C_X = \emptyset$. Furthermore, if $v_1, v_t \notin I_X$, then the following properties hold.

- *$P_X = P[V(P) \cap \beta(\mathbb{K}_X)]$ is an induced path in G between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$.*
- *$P_X - (B_\ell \cup B_r)$ is an induced path in $G[I_X]$.*

Proof. Consider a vertex $v \in (V(P) \cap I_X) \setminus \{v_1, v_t\}$, and let v_{-1} and v_{+1} be its two neighbors in P. Recall that $N(M_P) \cap \beta(\mathbb{K}_X) \subseteq C_X$, and hence $N_G(v) \cap M_P = \emptyset$. We thus observe, because $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup M_A$ and $V(P) \cap M_A = \emptyset$, it follows that the vertices v_{-1} and v_{+1} must belong to $\beta(\mathbb{K}_X)$. Furthermore, C_X is a clique, and for any $w \in C_X$ we have $N_G(v) \subseteq N_G(w)$ because $\mathbb{K}_X$ is not a clique in G. Here, we relied on Observation 6.13. Therefore, $V(P) \cap C_X = \emptyset$. Indeed, it it were not the case, then we obtain a chord in the induced path P between a vertex $w \in V(P) \cap C_X$ and (at least) one of v_{-1} or v_{+1} due to the containment $\{v_{-1}, v_{+1}\} \subseteq N_G(v) \subseteq N_G(w)$. This concludes the proof of the first part of the lemma.

Now, we turn to prove the second part of the lemma. Towards this, consider the set $V(P) \cap \beta(\mathbb{K}_X)$, and let $v_s \in I_X$ be the vertex with the smallest index (i.e., subscript) in P that belongs to the set I_X. The existence of such a vertex v_s follows from the assumption that

$(V(P) \setminus \{v_1, v_t\}) \cap I_X \neq \emptyset$. Moreover, note that $s \in \{2, \dots, t-1\}$ due to the assumption that $v_1, v_t \notin I_X$. Let v_e (possibly same as v_s) be the vertex with the largest index in P that belongs to I_X such that for every $i \in \{s, s+1, \dots, e\}$, $v_i \in I_X$. As before, we have that $e \in \{2, \dots, t-1\}$.

Next, we consider the vertices v_{s-1} and v_{e+1} along with the induced subpath $P' = P[\{v_{s-1}, v_s, \dots, v_{e+1}\}]$. From the construction of v_{s-1} and v_{e+1}, the assumption that $V(P) \cap M_A = \emptyset$, and the first part of the lemma, it follows that $v_{s-1}, v_{e+1} \notin I_X \cup M_A \cup C_X$. Moreover, $(v_{s-1}, v_s), (v_e, v_{e+1}) \in E(G)$, and for $v^* \in \{v_s, v_e\}$, we have $N_G(v^*) \subseteq \beta(\mathbb{K}_X) \cup M_A$. Therefore, $v_{s-1}, v_{e+1} \in (B_\ell \cup B_r) \setminus C_X$. Without loss of generality, we assume that $v_{s-1} \in B_\ell \setminus C_Q$. Then, $v_{e+1} \notin B_\ell \setminus C_Q$, since otherwise we have the chord (v_{s-1}, v_{e+1}) in P. This implies that $v_{e+1} \in B_r \setminus C_X$. Therefore, $P' = P[\{v_{s-1}, v_s, \dots, v_{e+1}\}]$ is an induced path from a vertex in $B_\ell \setminus C_X$ to a vertex in $B_r \setminus C_X$.

Notice that v_{s-1-i}, for any $i \geq 2$, cannot belong to B_ℓ, since otherwise there will be a chord in P (between v_{s-1-i} and v_{s-1}). We note that v_{s-2} could possibly belong to $B_\ell \setminus C_X$ but not to C_X. Symmetrically, we derive that v_{e+1+i}, for any $i \geq 2$, cannot belong to B_r, while v_{e+2} could possible belong to $B_r \setminus C_X$ but not to C_X. Let $s^* \in \{s-1, s-2\}$ be the smallest index such that $v_{s^*} \in V(P) \cap (B_\ell \setminus C_X)$, and $e^* \in \{e+1, e+2\}$ be the largest index such that $v_{e^*} \in V(P) \cap (B_r \setminus C_X)$. From this, we conclude that $P^* = P[\{v_{s^*}, v_{s^*+1}, \dots, v_{e^*}\}]$ is an induced path from a vertex in $B_\ell \setminus C_X$ to a vertex in $B_r \setminus C_X$.

Thus, to complete the proof of the lemma, it remains to show that $v_i \notin I_X$ for all $i \in [s-2] \cup \{e+2, e+3, \dots, t\}$. Suppose not, then there is an integer $i^* \in [s-2] \cup \{e+2, e+3, \dots, t\}$ such that $v_{i^*} \in I_X$. Since $v_{i^*} \in I_X$, it must hold that v_{i^*} belong to a bag, say B^* in $\mathbb{K}_X$ which is different from B_ℓ and B_r. Recall that P' is a sub-path of P from $v_{s-1} \in B_\ell \setminus C_X$ to $v_{e+1} \in B_r \setminus C_X$. Therefore, P' intersects every bag in the manageable clique path $\mathbb{K}_X$. In particular, it contains a vertex different from v_{i^*}, say v', from B^*. But then $(v', v_{i^*}) \in E(G)$ is a chord in the induced path P, which is a contradiction. This concludes the proof of the lemma. $\square$

Observation 6.16. *Let $v \in \beta(\mathbb{K}_X) \setminus C_X$. Then, v is not a center vertex of any AW in G that is not covered by $\mathcal{W}$.*

Proof. Let $\mathbb{O}$ be an AW in G that is not covered by $\mathcal{W}$, and suppose that $v \in \beta(\mathbb{K}_X) \setminus C_X$ is a center vertex of $\mathbb{O}$. Then, v must be adjacent (in G) to all the vertices of $\mathsf{base}(\mathbb{O})$. As M is a 9-redundant solution, there are at least five vertices of M in $\mathsf{base}(\mathbb{O})$, and therefore there are vertices $m_1, m_2 \in M$ such that $(m_1, m_2) \notin E(G)$ and $(m_1, v), (m_2, v) \in E(G)$. Moreover, from Observation 6.11, for (distinct) $u, w \in N_G(v) \cap M$ one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds. But $(m_1, m_2) \notin E(G)$, therefore, $\{m_1, m_2\} \in \mathcal{W}$ must hold. This contradicts the fact that $\mathbb{O}$ is not covered by $\mathcal{W}$. $\square$

Let $\mathbb{O}$ be an AW (not covered by $\mathcal{W}$) in G. Recall that $P(\mathbb{O})$ denotes the extended base of $\mathbb{O}$ (including base vertices, t_ℓ and t_r). We have the following notion of distance of a vertex in $\mathbb{K}_X$ from the end bags B_ℓ and B_r. We use this notion in marking bags that satisfy certain properties and are closest to the endpoints of $\mathbb{K}_X$. In this context, recall that we have an ordering of the bags from left to right, where B_ℓ is the leftmost bag and B_r is the rightmost bag in $\mathbb{K}_X$.

Definition 6.2. Let $v \in I_X$. The *distance between v and B_ℓ* is defined as the number of bags between B_ℓ and the right-most bag in $\mathbb{K}_X$ that contains v. Symmetrically, the *distance between v and B_r* is defined as the number of bags between between B_r and the left-most bag in $\mathbb{K}_X$ that contains v.

Towards Our Case Distinction. In what follows, we consider two cases based on the intersection between the vertex set of $\mathbb{O}$ and $I_X \cup M_A$. Before this, for the sake of clarity, let us recall a few facts. First, $\mathbb{O}$ is an AW in G that is not covered by $\mathcal{W}$. Second, $\mathbb{K}_X$ is a manageable clique path which does not induce a clique in G. The sets B_ℓ and B_r are cliques in G, and $B_\ell \cup B_r \cup M_A$ separate I_X from the rest of the graph. Furthermore, every vertex of M_A is adjacent to all vertices in $\beta(\mathbb{K}_X)$ in G (by Observation 6.13). The vertices of $\beta(\mathbb{K}_X) \setminus C_X$, and in particular I_X, cannot be the center vertices of any AW in G that is not covered by $\mathcal{W}$ (by Observation 6.16). Therefore, every vertex of I_X is either a base vertex or a terminal of $\mathbb{O}$.

6.2.1 $V(\mathsf{base}(\mathbb{O})) \cap I_X = \emptyset$ or $V(P(\mathbb{O})) \cap M_A \neq \emptyset$

Irrespective of whether $V(\mathsf{base}(\mathbb{O})) \cap I_X = \emptyset$ or $V(P(\mathbb{O})) \cap M_A \neq \emptyset$, let us first observe that since $\mathbb{O}$ is an AW, for any clique A in G, we have $|V(A) \cap V(\mathbb{O})| \leq 4$. This implies that $|V(\mathbb{O}) \cap (B_\ell \cup B_r)| \leq 8$. Moreover, since $\mathbb{O}$ is not covered by $\mathcal{W}$, for distinct $m, m' \in M_A \cap V(\mathbb{O})$, we have $(m, m') \in E(G)$. Thus, $|V(\mathbb{O}) \cap M_A| \leq 4$. From this, we obtain the following inequality:

$$|V(\mathbb{O}) \cap (M_A \cup B_\ell \cup B_r)| \leq 12.$$

Let c_1, c_2 be the center vertices of $\mathbb{O}$ (in the case of a †-AW, we have $c = c_1 = c_2$). Then, depending on whether $V(\mathsf{base}(\mathbb{O})) \cap I_X = \emptyset$ or $V(P(\mathbb{O})) \cap M_A \neq \emptyset$, we note the following.

- First, suppose that $V(\mathsf{base}(\mathbb{O})) \cap I_X = \emptyset$. In this subcase, $V(\mathbb{O}) \cap I_X \subseteq \{t_\ell, t_r, t\}$ (due to Observation 6.16).
- Second, suppose that there is a vertex $m \in V(\mathsf{base}(\mathbb{O})) \cap M_A$. Recall that every vertex in M_A is adjacent to all the vertices in I_X. Thus, in this subcase, $|V(\mathbb{O}) \cap I_X| \leq 2$, otherwise $m \in V(\mathsf{base}(\mathbb{O}))$ will be adjacent to three vertices of $V(\mathbb{O}) \setminus \{c_1, c_2\}$.

In summary, $V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \cup M_A)$ contains at most 15 vertices: up to 12 of these vertices are in $M_A \cup B_\ell \cup B_r$, and up to 3 of these vertices are in I_X. We will use these bounds to derive our next marking scheme. In particular, since we deal with an obstruction whose intersection with $\beta(\mathbb{K}_X) \cup M_A$ is upper bounded by a fixed constant, the relevance of the tool of representative families (defined in Section 2) is presented as a possibility—intuitively, we would like to capture enough vertices to represent every possibility of how the (up to) 3 vertices from I_X can "behave" within the small intersection. Towards that end, we proceed as follows.

Computation of representative families. Consider a tuple $\mathcal{R} = (R, R_B, R_I)$ where R is a graph on the vertex set $R_B \cup R_I$ (these are new dummy vertices), $|R_B| \leq 12$ and $|R_I| \leq 3$. Furthermore, consider a set $Z \subseteq M_A \cup B_\ell \cup B_r$ of $|R_B|$ vertices, a bijective function $f : Z \to R_B$, and an integer $d \in [3]$. For every such tuple (R, Z, f, d), we will perform a computation of a representative family as follows. Here, the family to be represented is $\mathcal{A}_{R,Z,f,d}$, the family of all d-sized subsets $Y \subseteq I_X$ such that the following condition is satisfied.

> There exists an isomorphism φ between $G[Z \cup Y]$ and R whose restriction to Z is equal to f, that is, for all $z \in Z$ we have $\varphi(z) = f(z)$.

Intuitively, we consider every "frame" that consists of the following: *(i)* the identity and topology of the (up to) 12 vertices in $M_A \cup B_\ell \cup B_r$ that lie in the intersection—this includes the specification of what are the identities of these vertices (given by Z) and what are the edges among them in G (given by $R[R_B]$); *(ii)* the topology of the (up to) 3 vertices in X_I that lie in the intersection (given by $R[R_I]$) and the edges between them and the previously mentioned 12 vertices (given by R). However, this information is not sufficient, and we require to also have explicit restriction of which vertex in Z is mapped to which vertex in R, and this is provided to us by the function f.

Next, consider the matroid $\mathcal{M} = (U, \mathcal{I})$ where $U = V(G)$ and $\mathcal{I} = \{U' \subseteq U \mid |U'| \leq d+k+2\}$. Notice that $\mathcal{M}$ is a uniform matroid, and therefore is representable over a field of size at least $d + k + 2$ [50]. Thus, using Theorem 2 we obtain a $(k+2)$-representative family $\widehat{\mathcal{A}_{R,Z,f,d}} \subseteq_{rep}^{k+2} \mathcal{A}_{R,Z,f,d}$.

Marking based on the representative families. We now construct a set $\mathbb{K}(\text{Rep}, \mathbb{K}_X)$ of bags in $\mathbb{K}_X$ as follows. For every tuple (R, Z, f, d) defined above for the manageable clique path $\mathbb{K}_X$, and for every vertex v that belongs to at least one set in $\widehat{\mathcal{A}_{R,Z,f,d}}$, we choose (arbitrarily) a bag in $\mathbb{K}_X$ that contains v, and add this bag to the set $\mathbb{K}(\text{Rep}, \mathbb{K}_X)$. Finally, we let $\mathbb{K}(\text{Rep})$ be the union of the bags in $\mathbb{K}(\text{Rep}, \mathbb{K}_Y)$ across every $\mathbb{K}_Y$ that is a manageable clique path which does not induce a clique in G.

Marking Scheme V. Add all the bags in $\mathbb{K}(\text{Rep})$ to $\mathcal{B}$.

Towards bounding the number of bags we marked using the above marking scheme, consider a manageable clique path $\mathbb{K}_X$ with end bags B_ℓ, B_r, which does not induce a clique in G. We observe that there are at most $\mathcal{O}(1)$ choices for the graph R and its partition into R_B and R_I. Furthermore, there are at most $\binom{|M_A \cup B_\ell \cup B_r|}{\leq 12}$ choices for Z, and at most $\mathcal{O}(1)$ choices for f given the choice of Z. Thus, by Theorem 2 there are at most $\mathcal{O}((k+2)^3)$ sets in $\widehat{\mathcal{A}_{R,Z,f,d}}$ and each set contains at most $d \leq 3$ vertices. Hence, overall we marked at most $\mathcal{O}((2\eta + |M|)^{12}(k+2)^3)$ bags in the manageable clique path $\mathbb{K}_X$. As there are at most $\mathcal{O}(\eta^3|M|)$ manageable clique paths in K, Marking Scheme V marks at most $\boxed{\mathcal{O}(\eta^{15}|M|k^3)}$ bags.

In the following, we prove a property regarding bags marked by Marking Scheme V.

Lemma 6.17. *Let S be a set of size at most $k+2$ that intersects every set in $\mathcal{W}$, and $\mathbb{O}$ be an AW in $G - S$ such that $V(\mathsf{base}(\mathbb{O})) \cap I_X = \emptyset$ or $V(P(\mathbb{O})) \cap M_A \neq \emptyset$. Then, there is also an AW $\mathbb{O}'$ in $G - S$ such that $\mathbb{O}'$ is $\mathbb{K}_X$-manageable, and $\mathbb{O}' - I_X = \mathbb{O} - I_X$.*

Proof. Consider the graph $R = \mathbb{O}[V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \cup M_A)]$ (where we forget the "labelling" of the vertices, i.e., the graph R is supposed to be on $|V(R)|$ dummy vertices). Let $Z = V(R) \cap (M_A \cup B_\ell \cup B_r)$ and $Y = V(R) \setminus Z$. From the earlier discussion in this subsection, it follows that $|V(R)| \leq 15$, $|Z| \leq 12$, and $|Y| \leq 3$. Let $d = |Y|$. Moreover, f is the function that maps every vertex in Z to the vertex in R that was originally labeled by Z.

Notice that $Y \in \mathcal{A}_{R,Z,f,d}$. Thus, from Theorem 2 there is a set $Y' \in \widehat{\mathcal{A}_{R,Z,f,d}}$ such that the following condition holds.

> There is an isomorphism φ between $G[Z \cup Y']$ and R whose restriction to Z is equal to f.

Since both Y and Y' are subsets of I_X, their neighbors in G belong to $M_A \cup B_\ell \cup B_r \cup I_X$. Thus, both $N(Y) \cap V(\mathbb{O}) \subseteq Z$ and $N(Y') \cap V(\mathbb{O}') \subseteq Z$. Together with the condition above, we thus obtain that $\mathbb{O}' = G[V(\mathbb{O} - Y) \cup Y']$ is isomorphic to $\mathbb{O}$. Hence $\mathbb{O}'$ is an AW in $G - S$ with the property that all of the vertices of $\mathbb{O}'$ from $\mathbb{K}_X$ appear in the marked bags of $\mathbb{K}_X$. This means that $\mathbb{O}'$ is $\mathbb{K}_X$-manageable. Finally observe that, by construction, $\mathbb{O}' - I_X = \mathbb{O} - I_X$. This concludes the proof of the lemma. □

In the following subsection, we consider the problem of obtaining $\mathbb{K}_X$-manageable obstructions when $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$. We note that in that subsection, we treat the bags marked by Marking Scheme V as unmarked and only consider the bags marked by Marking Schemes I,II, III, and IV as marked bags.

6.2.2 $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$

The goal of this subsection will be to show that any AW $\mathbb{O}$ in G that is not covered by $\mathcal{W}$ is, in fact, already a $\mathbb{K}_X$-manageable obstruction. To this end, we let $\mathbb{O}$ be an AW in G. Furthermore, we remind that c_1 and c_2 are the centers of $\mathbb{O}$ (in case $\mathbb{O}$ is a †-AW, we have $c = c_1 = c_2$), t_ℓ, t_r

are the non-shallow terminals, t is the shallow terminal, $\mathsf{base}(\mathbb{O})$ is the base, and $P(\mathbb{O})$ is the extended base.

In the following, we obtain some useful properties of $\mathbb{O}$ that satisfy the premise of this subsection, i.e. $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$. This will be done in a sequence of four statements, after which we will be able to obtain the desired result. We first observe that the center(s) must belong to $C_X \cup M_A$.

Observation 6.18. *If $\mathbb{O}$ is an AW not covered by $\mathcal{W}$ and $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$, then $c_1, c_2 \in C_X \cup M_A$.*

Proof. Consider $v \in V(\mathsf{base}(\mathbb{O})) \cap I_X$. Because $v \in I_X$, we have $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup M_A$, hence no vertex outside $\beta(\mathbb{K}_X) \cup M_A$ can be a center (as c_1, c_2 must belong to $N_G(v)$). Moreover, recall that by Observation 6.16, no vertex in $\beta(\mathbb{K}_X) \setminus C_X$ can be a center vertex of an AW in G (that is uncovered by $\mathcal{W}$). Therefore, we have that $c_1, c_2 \in M_A \cup C_Q$. $\square$

Secondly, we observe that the non-shallow terminals do not belong to $\beta(\mathbb{K}_X) \cup M_A$ (which already brings us close to the goal of this section), the base does not traverse C_X, and the shallow terminal does not belong to $C_X \cup M_A$.

Observation 6.19. *If $\mathbb{O}$ is an AW not covered by $\mathcal{W}$, $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$, then $t_\ell, t_r \notin \beta(\mathbb{K}_X) \cup M_A$. Furthermore, $V(\mathsf{base}(\mathbb{O})) \cap (C_X \cup M_A) = \emptyset$ and $t \notin C_X \cup M_A$.*

Proof. From Observation 6.18, $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ implies that $c_1, c_2 \in C_X \cup M_A$. From Observation 6.13 we have that any vertex of $C_X \cup M_A$ is adjacent to every vertex in $\beta(\mathbb{K}_X)$ in G. As $c_1 \in C_X \cup M_A$, is not adjacent to t_ℓ, we obtain that $t_\ell \notin \beta(\mathbb{K}_X)$. Towards a contradiction, consider the case where $t_\ell \in M_A$. Since $\mathbb{O}$ is not covered by $\mathcal{W}$, we have $\{c_1, t_\ell\} \notin \mathcal{W}$. But then from Observation 6.13 we obtain that $(c_1, t_\ell) \in E(G)$. This contradicts that $\mathbb{O}$ is an AW in G. From the above we obtain that $t_\ell \notin \beta(\mathbb{K}_X) \cup M_A$. An analogous argument can be given to show that $t_r \notin \beta(\mathbb{K}_X) \cup M_A$. This proves the first part of the observation.

Next, towards a contradiction, suppose that there exists $w \in V(\mathsf{base}(\mathbb{O})) \cap (C_X \cup M_A)$. By the assumption that $V(P(\mathbb{O})) \cap M_A = \emptyset$, we have $w \notin M_A$. Hence $w \in C_X$, which means (by Observation 6.13) that w is adjacent to every vertex in $\beta(\mathbb{K}_X) \cup M_A$. Let $v \in V(\mathsf{base}(\mathbb{O})) \cap I_X$ (which exists by the assumption that $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$) and u be the neighbor of v in $P(\mathbb{O})$ that is different than w. Recall that $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup M_A$, therefore $u \in \beta(\mathbb{K}_X) \cup M_A$. However, this implies that $P(\mathbb{O})[\{v, u, w\}]$ is a cycle on three vertices, contradicting that $P(\mathbb{O})$ is an induced path.

Finally, if $t \in C_X \cup M_A$, then $(t, v) \in E(G)$ ($\mathbb{O}$ is not covered by $\mathcal{W}$), which is a contradiction. This completes the proof. $\square$

Third, we consider the subpath $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) - C_X]$ of $P(\mathbb{O})$. Due to Lemma 6.15, the following lemma is almost immediate.

Lemma 6.20. *If $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$, then $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) \setminus C_X]$ is an induced path between a vertex in $B_\ell \setminus C_X$ and $B_r \setminus C_X$, and P_X is a subpath of $\mathsf{base}(\mathbb{O})$.*

Proof. We note that $P(\mathbb{O})$ is an induced path in G and $\mathbb{O}$ is not covered by $\mathcal{W}$. We further note that the following conditions are satisfied.

1. $(V(P(\mathbb{O})) \setminus \{v_1, v_t\}) \cap I_X \neq \emptyset$. This follows from our assumption that $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$.

2. $V(P(\mathbb{O})) \cap M_A = \emptyset$, as this is one of our assumptions.

3. $V(P(\mathbb{O})) \cap (V(G) \setminus I_X) \neq \emptyset$ and $t_\ell, t_r \notin I_X$. This follows from the fact that $t_\ell, t_r \notin \beta(\mathbb{K}_X) \cup M_A$, which is obtained from Observation 6.19

Thus, using Lemma 6.15, we obtain that $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) - C_X]$ is an induced path between a vertex in $B_\ell \setminus C_X$ and $B_r \setminus C_X$ and P_X is a subpath of $\mathsf{base}(\mathbb{O})$. $\square$

Using Lemma 6.20, we obtain the following observation.

Observation 6.21. *If $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$, then $t \notin \beta(\mathbb{K}_X) \cup M_A$.*

Proof. First, towards a contradiction, consider the case where $t \in M_A$. Let $v \in V(\mathsf{base}(\mathbb{O})) \cap I_X$, which exists by our assumption. But from Observation 6.13 we have that $(t, v) \in E(G)$, which contradicts that $\mathbb{O}$ is an AW in G. Second, towards a contradiction, consider the case where $t \in \beta(\mathbb{K}_X)$. Using Lemma 6.20, we obtain that $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) - C_X]$ is an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$, and P_X is a subpath of $\mathsf{base}(\mathbb{O})$. But then P_X intersects every bag in $\mathbb{K}_X$ and t must lie in one of the bags in $\mathbb{K}_X$. From this we conclude that there is $v \in V(P(\mathbb{O}))$ such that $(t, v) \in E(G)$, which again contradicts that $\mathbb{O}$ is an AW in G. $\square$

The next lemma, whose proof was the goal of this subsection, follows directly from the above results and the definition of $\mathbb{K}_X$-manageable obstructions. Indeed, Observation 6.19 states that the non-shallow terminals cannot belong to $\beta(\mathbb{K}_X)$, and Observation 6.21 states that the shallow terminal cannot belong to $\beta(\mathbb{K}_X)$.

Lemma 6.22. *Let $\mathbb{O}$ be an AW in G such that $\mathbb{O}$ is not covered by $\mathcal{W}$, $V(\mathsf{base}(\mathbb{O})) \cap I_X \neq \emptyset$, and $V(P(\mathbb{O})) \cap M_A = \emptyset$. Then $\mathbb{O}$ is a $\mathbb{K}_X$-manageable obstruction.*

6.3 Nice-clique Paths and Nice-obstructions

We now consider a pair of consecutive marked bags in $\mathbb{K}$ that were marked by Marking Schemes I to V. In particular, for each manageable clique path $\mathbb{K}_{X'}$ we marked a collection of bags in $\mathbb{K}_{X'}$ via Marking Scheme V, which (further) partitions $\mathbb{K}_{X'}$ into sub-clique paths that will be called *nice-clique paths*. Formally, a *nice-clique path* is a sub-clique path $\mathbb{K}[B_\ell, B_r]$ such that B_ℓ and B_r are consecutive marked bags. We note that a nice-clique path does not induce a clique in G, since it contains at least two distinct bags of $\mathbb{K}$. Furthermore, any nice-clique path is contained in a manageable clique path, and therefore it also has the properties of a manageable clique path. Now, for any nice-clique path $\mathbb{K}_X$ that is contained in a manageable clique path $\mathbb{K}_{X'}$, we define the sets $B_\ell, B_r, I_X, C_X, M_A$ and M_P in the same way as before. We note that the sets M_A and M_P partition M in exactly the same way as $M_{A'}$ and $M_{P'}$ (the partition defined with respect to $\mathbb{K}_{X'}$) partitions M (see Observation 6.13 and the definitions of these sets). Furthermore, $C_{X'} \subseteq C_X$, since $\mathbb{K}_X$ is a sub-clique path of $\mathbb{K}_{X'}$. In the following we define the notion of *nice-obstructions.*

Definition 6.3. Let $\mathbb{K}_X$ be a nice-clique path with B_ℓ and B_r as the first and last bags, respectively, and let $J = V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X)$. A manageable obstruction $\mathbb{O}$ is called a $\mathbb{K}_X$*-nice-obstruction*, if one of the following holds.

1. $J \subseteq B_\ell \cup B_r$, or
2. $\mathbb{O}[J]$ is an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$, such that $(V(\mathbb{O}) \cap \beta(\mathbb{K}_X)) \setminus (B_\ell \cup B_r) \subseteq I_X$.

A manageable obstruction $\mathbb{O}$ is called a *nice-obstruction* if for every nice-clique path $\mathbb{K}_X$, it is $\mathbb{K}_X$-nice.

The following lemma shows that an induced cycle on at least 4 vertices, which is not covered by $\mathcal{W}$, is always a nice-obstruction. We note that by definition, a chordless cycle on 4 vertices is a manageable obstruction.

Lemma 6.23. *Let $\mathbb{O}$ be a chordless cycle on at least 4 vertices that is not covered by $\mathcal{W}$. Then $\mathbb{O}$ is a nice-obstruction.*

Proof. Let us consider a nice-clique path $\mathbb{K}_X$ and suppose that $J = V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X) \not\subseteq B_\ell \cup B_r$. Consider a vertex $v \in J \setminus (B_\ell \cup B_r)$. Since $X_I = \beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$, we have that $v \in J \cap X_I$. Therefore, there is a pair of (distinct) vertices $m_1, m_2 \in M$ such that the path segment P between m_1 and m_2 in $\mathbb{O}$ contains the vertex v and $V(\mathbb{O}) \setminus V(P) \neq \emptyset$. Here, we rely on the fact that $\mathbb{O}$ is not covered by $\mathcal{W}$, therefore, $|M \cap V(\mathbb{O})| \geq 10$, which implies that $|V(\mathbb{O})| \geq 10$. Let P^* be the sub-path of P that contains v, such that $|V(P^*) \cap M| = 2$. Note that P^* exists, and could possibly be same as P. Let P^* be the path from $m_1^* \in M$ to $m_2^* \in M$. Next, we argue that $m_1^*, m_2^* \notin M_A$. Consider the case when both $m_1^*, m_2^* \in M_A$. Since $\mathbb{O}$ is not covered by $\mathcal{W}$, from Observation 6.13, we have $(m_1^*, m_2^*) \in E(G)$. But this contradicts that P^* is an induced path in G. Next, suppose that $m_1^* \in M_A$, $m_2^* \in M_P$ (the other case is symmetric). But then, we have that $(v, m_2^*) \notin E(G)$. Observe that v has no neighbor outside $\beta(\mathbb{K}_X) \cup M_A$ and m_1 is adjacent to all vertices in $\beta(\mathbb{K}_X) \cup (M_A \cap V(\mathbb{O}))$ (Observation 6.13). Now let u be the neighbor of v in the sub-path of P^* from v to m_2. Observe that $u \in \beta(\mathbb{K}_X)$, and therefore we obtain a chord (m_1, u) in P^*, which is a contradiction. Therefore, $m_1^*, m_2^* \notin M_A$, and thus we have that $V(P^*) \cap M_A = \emptyset$. Observe that P^* satisfies the premise of Lemma 6.15, as the endpoints of P^* lie outside $\beta(\mathbb{K}_X)$, it contains an internal vertex from I_X, and $V(P) \cap M_A = \emptyset$. Therefore, $P^*[V(P^*) \cap \beta(\mathbb{K}_X)]$ is an induced path from a vertex in $B_\ell \setminus C_X$ to a vertex in $B_r \setminus C_X$ such that $P^* - (B_\ell \cup B_r)$ is an induced path contained in I_X. Therefore $\mathbb{O}$ is $\mathbb{K}_X$-nice-obstruction. Finally, as this argument holds for every nice-clique path $\mathbb{K}_X$, the lemma follows. $\square$

Next, for each obstruction (not covered by $\mathcal{W}$) we argue about existence of a nice-obstruction.

Lemma 6.24. *Let $S \subseteq V(G)$ be a set of size at most $k+2$ which intersects each set in $\mathcal{W}$. If $\mathbb{O}$ is an obstruction in $G - S$ that is not covered by $\mathcal{W}$, then there is a nice-obstruction $\mathbb{O}'$ in $G - S$ (that is not covered by $\mathcal{W}$).*

Proof. Since S intersects each set in $\mathcal{W}$, $\mathbb{O}$ contains at least 10 vertices from M. If $\mathbb{O}$ is a chordless cycle, then by Lemma 6.23, it is a nice-obstruction. Otherwise $\mathbb{O}$ is an AW, and suppose that it is not a nice-obstruction. Let $\mathbb{O}'$ be an obstruction in $G - S$ that is $\mathbb{K}_X$-manageable for every manageable path $\mathbb{K}_X$. It is obtained by iteratively applying Lemma 6.17 or Lemma 6.22 for every manageable clique path $\mathbb{K}_X$, depending on the sets $V(\mathsf{base}(\mathbb{O})) \cap I_X$ and $V(P(\mathbb{O})) \cap M_A$. Note that, each application of these lemmas modifies the obstruction only within the corresponding manageable clique path. Thus we have that $\mathbb{O}'$ is not covered by $\mathcal{W}$ and $\mathbb{O}'$ is a manageable obstruction.

We claim that $\mathbb{O}'$ is a nice-obstruction in $G - S$. Consider a manageable clique path $\mathbb{K}_Y$, and a nice-clique path $\mathbb{K}_X = \mathbb{K}_Y[B_\ell, B_r]$ (i.e. it is nice-clique path that is contained in the manageable clique path $\mathbb{K}_Y$, and it the sub-clique path between a pair of consecutive marked bags B_ℓ, B_r in $\mathbb{K}_Y$). We must show that $V(\mathbb{O}') \cap (\beta(\mathbb{K}_X) \setminus C_X)$ is either a subset of $B_\ell \cup B_r$, or an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$ such that $(V(\mathbb{O}) \cap \beta(\mathbb{K}_X)) \setminus (B_\ell \cup B_r) \subseteq I_X$, where $C_X = B_\ell \cap B_r$. That is, we show that $\mathbb{O}$ is $\mathbb{K}_X$-nice. We note that the partition of M into M_A and M_P that be obtain with respect to $\mathbb{K}_Y$ is same as the partition that we obtain with respect to $\mathbb{K}_X$. Thus we deal with the partition of M into M_A and M_P which is defined by $\mathbb{K}_Y$. Next, we consider the following cases.

1. Consider the case when $V(\mathsf{base}(\mathbb{O})) \cap I_Y = \emptyset$. In this case, by Lemma 6.17, $\mathbb{O}'$ is $\mathbb{K}_Y$-manageable. Thus, the terminals of $\mathbb{O}'$ must lie in the marked bags of $\mathbb{K}_Y$, and hence they cannot belong to vertices in $\beta(\mathbb{K}_X) \setminus C_X$. Next, we give argument for the center vertices. Assume that (at least) one of the centers say c_1 is in $\beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$. The vertex c_1 must be adjacent to each vertex in $V(\mathsf{base}(\mathbb{O}))$. Since $\mathbb{O}'$ is not covered by $\mathcal{W}$, it must have at

least 5 vertices in $M \cap V(\mathbb{O}')$. Consider two non-adjacent vertices $m, m' \in M \cap V(\mathsf{base}(\mathbb{O}))$. Since the only neighbors of c_1 in M are vertices in M_A, we have $m, m' \in M_A$. But then from Observation 6.13 we have that $(m, m') \in E(G)$, which is a contradiction. From the above discussions, together with the assumption that $V(\mathsf{base}(\mathbb{O})) \cap I_Y = \emptyset$ and the fact that $I_X \subseteq I_Y$, we conclude that $V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X) \subseteq B_\ell \cup B_r$.

2. Consider the case when $V(P(\mathbb{O})) \cap M_A \neq \emptyset$. Similar to the previous case, in this case $\mathbb{O}'$ is $\mathbb{K}_Y$-manageable, from Lemma 6.17. Thus, the terminals of $\mathbb{O}'$ must lie in the marked bags of $\mathbb{K}_Y$, and hence they cannot belong to vertices in $\beta(\mathbb{K}_X) \setminus C_X$. By using an arguments similar to the one used for the previous case, we can deduce that the centres cannot belong to $\beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$. Finally, if there is $v \in \beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$ in $V(\mathsf{base}(\mathbb{O}))$, consider its two (non-adjacent) neighbors x, y in $\mathsf{base}(\mathbb{O})$. Notice that since $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup M_A$, $x, y \in \beta(\mathbb{K}_X) \cup M_A$. Since $\mathbb{O}'$ is not covered by $\mathcal{W}$, using Observation 6.13 we can deduce that at most one of x, y can belong to M_A. If $x \in M_A$ and $y \notin M_A$, which means that $y \in \beta(\mathbb{K}_X)$, and then using Observation 6.13, we have that $(x, y) \in E(G)$. Otherwise, $x, y \in \beta(\mathbb{K}_X)$. Consider $u \in V(P(\mathbb{O})) \cap M_A$, which exists by our assumption. But then u, v are both x, y, contradicting that $P(\mathbb{O})$ is an induced path. Thus, we conclude that $V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X) \subseteq B_\ell \cup B_r$.

3. Otherwise, we have $\mathsf{base}(V(\mathbb{O})) \cap I_Y \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$. Then by Lemma 6.22, $\mathbb{O}'$ is $\mathbb{K}_Y$-manageable for the manageable clique path $\mathbb{K}_Y$. Therefore, from Lemma 6.20 we know that $P = \mathbb{O}'[V(\mathbb{O}') \setminus (\beta(\mathbb{K}_Y) - C_Y)$ is an induced path from a vertex in $B'_\ell \setminus C_Y$ to a vertex in $B'_r \setminus C_Y$, where B'_ℓ and B'_r are the first and last bags of $\mathbb{K}_Y$ respectively, and $C_Y = B'_\ell \cap B'_r$. Observe that P visits every bag in $\mathbb{K}_X = \mathbb{K}_Y[B_\ell, B_r]$. Moreover, P is a subpath of $\mathsf{base}(\mathbb{O}')$ and by Observation 6.18 $\{c_1, c_2\} \subseteq C_Y \cup M_A$. And Observation 6.19 implies that $t_\ell, t_r \notin \beta(\mathbb{K}_Y)$ (the endpoints of $P(\mathbb{O})$). Also note that $C_Y \subseteq C_X$ by definition. Now, if $V(P) \cap (\beta(\mathbb{K}_X) \setminus C_X) \subseteq B_\ell \cup B_r$, then $\mathbb{O}'$ is $\mathbb{K}_X$-nice. Otherwise, $V(P) \cap I_X \neq \emptyset$. Now observe that the path $P(\mathbb{O})$ and $\mathbb{K}_X$ satisfy the following conditions: (i) $P(\mathbb{O})$ contains has an internal vertex from I_X, (ii) $V(P(\mathbb{O})) \cap M_A = \emptyset$, (iii) $V(P(\mathbb{O})) \cap (V(G) \setminus I_X) \neq \emptyset$ and (iv) the endpoints of $P(\mathbb{O})$ lie outside I_X. Thus, $P(\mathbb{O})$ satisfies the premise of Lemma 6.15. Therefore, $P(\mathbb{O})[V(P(\mathbb{O})) \cap (\beta(\mathbb{K}_X) \setminus C_X)$ is an induced path from a vertex in $B_\ell \setminus C_X$ to a vertex in $B_r \setminus C_X$ such that $(V(P(\mathbb{O}')) \cap \beta(\mathbb{K}_X)) \setminus (B_\ell \cup B_r) \subseteq I_X$. Hence $V(\mathbb{O}') \cap (\beta(\mathbb{K}_X) \setminus C_X)$ is an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$ such that $(V(\mathbb{O}) \cap \beta(\mathbb{K}_X)) \setminus (B_\ell \cup B_r) \subseteq I_X$. Thus, $\mathbb{O}$ is $\mathbb{K}_X$-nice.

This concludes the proof. $\square$

Let us remark that the proof of Lemma 6.24, Definition 6.3 and earlier results show the following corollary.

Corollary 6.25. *If $\mathbb{O}$ is a nice-obstruction in G that is not covered by $\mathcal{W}$, then for any nice-clique path $\mathbb{K}_X$, $\mathbb{O} \cap \beta(\mathbb{K}_X)$ is either a subset of $B_\ell \cup B_r$, or $V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X)$ is an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$ that contains a vertex of I_X. Furthermore, in the second case and when $\mathbb{O}$ is an AW, $V(\mathbb{O}) \cap (C_X \cup M_A) = \{c_1, c_2\}$, and $(V(\mathbb{O}) \cap \beta(\mathbb{K}_X)) \setminus (B_\ell \cup B_r)$ is an induced path in I_X which is a sub-path of $\mathsf{base}(\mathbb{O})$. Here, B_ℓ and B_r are the first and last bags of $\mathbb{K}_X$, respectively.*

We will require a strengthening of the above corollary that allows us to "replace" the path $P = \mathbb{O}[V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X)]$ in $\mathbb{O}$ with another path P' between the endpoint bags of $\mathbb{K}_X$ and obtain a new obstruction. To obtain this property, we need to further partition a nice-clique path by marking the following collection of bags. We note that our next marking scheme is similar to Marking Scheme III, therefore, we use similar notations.

Consider a nice-clique path $\mathbb{K}_X$ with endpoint bags B_ℓ and B_r. Before moving on to our next marking scheme, we construct two sets of bags, $\mathcal{T}_1(X)$ and $\mathcal{T}_2(X)$. Initially, we have $\mathcal{T}_1(X) = \{B_\ell, B_r\}$. For each $u \in B_\ell$, let $\tilde{B}_u(X)$ be the first bag in $\mathbb{K}_X$ which does not contain u, where if such a bag does not exist, then we set $\tilde{B}_u(X) = B_r$. Additionally, for each $v \in B_r \setminus B_\ell$, let $\tilde{B}_v(X)$ to be the first bag in $\mathbb{K}_X$ which contains v, where if such a bag does not exists, then we set $\tilde{B}_v = B_r$. We add all the bags in $\{\tilde{B}_u(X) \mid u \in B_\ell\} \cup \{\tilde{B}_v(X) \mid v \in B_r \setminus B_\ell\}$ to $\mathcal{T}_1(X)$. We initialize $\mathcal{T}_2(X) = \mathcal{T}_1(X)$. Furthermore, for each bag $B \in \mathcal{T}_1(X)$ in $\mathbb{K}_X$, we add to $\mathcal{T}_2(X)$ the bags adjacent to B, namely B^{-1} and B^{+1} (if they exist) in $\mathbb{K}_X$. Note that the number of bags in $\mathcal{T}_2(X)$ is bounded by $\mathcal{O}(\eta)$. Finally, we let $\mathbb{K}(\mathcal{T})$ to be the union of the sets $\mathcal{T}_2(X)$ taken over all nice-clique paths $\mathbb{K}_X$.

Marking Scheme VI. Add all the bags in $\mathbb{K}(\mathcal{T})$ to $\mathcal{B}$.

We marked at most $\mathcal{O}(\eta)$ bags for each nice-clique path. Recall that we have at most $\mathcal{O}(\eta^3|M|)$ manageable clique paths and for each manageable clique path we marked at most $\mathcal{O}(\eta^{15}|M|k^3)$ bags in $\mathbb{K}$ using Marking Scheme IV and V, that partitioned the manageable clique path into nice-clique paths. Hence, in Marking Scheme VI we marked at most $\boxed{\mathcal{O}(\eta^{16}|M|k^3)}$ bags in $\mathbb{K}$.

Next, we state an observation regarding the region between pair of consecutive marked bags in a nice-clique path. We note that this observation is similar to Observation 6.10 presented in Section 6.1.2.

Observation 6.26. *Consider a pair B_p, B_q of consecutive marked bags in a nice-clique path $\mathbb{K}_X = \mathbb{K}[B_\ell, B_r]$, such that $\mathbb{K}[B_p, B_q]$ contains at least 3 bags. Then for any $B, B' \in \mathbb{K}[B_p, B_q]$, we have $B \cap (B_\ell \cup B_r) = B' \cap (B_\ell \cup B_r)$.*

Let us review the structural results we have obtained till now. Let $\mathbb{O}$ be a nice AW in G which is not covered by $\mathcal{W}$. Notice that the terminal vertices and center vertices of $\mathbb{O}$ either lie in marked bags of $\mathbb{K}$, or lie in M, or lie outside $\mathbb{K}$ (from Definition 6.3, Lemma 6.23 and Lemma 6.24). Let $\mathbb{K}_X$ be a nice-clique path such that $\mathbb{O}[V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X)]$ is an induced path P between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$ that contains a vertex in $\beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$. Here, B_ℓ and B_r are endpoint bags of $\mathbb{K}_X$. Note that the vertices in $\beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$ are unmarked vertices up till Marking Scheme V, and therefore the vertices in $P - (B_\ell \cup B_r)$ lie in $\mathsf{base}(\mathbb{O})$. From our arguments in Lemma 6.24, we have the following properties. As P contains an unmarked vertex (up till Marking Scheme V), $\mathsf{base}(\mathbb{O}) \cap I_X \neq \emptyset$ and $V(P(\mathbb{O})) \cap M_A = \emptyset$. The vertices of P lie in $P(\mathbb{O})$. Furthermore, the internal vertices of P lie in $\mathsf{base}(\mathbb{O})$ and $P - (B_\ell \cup B_r)$ is an induced path contained in I_X. The centers c_1, c_2 of $\mathbb{O}$ lie in $C_X \cup M_A$. The shallow terminal t of $\mathbb{O}$ lies outside $\beta(\mathbb{K}_X)$ as P intersects every bag in $\mathbb{K}_X$. Consider any pair of consecutive marked bags B_i, B_j in $\mathbb{K}_X$ under Marking Scheme VI. Let $\mathbb{K}_Y$ denote the sub-clique path $\mathbb{K}[B_i, B_j]$, and let C_Y denote the set $B_i \cap B_j$. Consider the path $P_Y = P[V(P) \cap \beta(\mathbb{K}_Y)]$ and suppose that P_Y contains a vertex in $\beta(\mathbb{K}_Y) \setminus C_Y$. Then, as P_Y is an induced path, it is disjoint from C_Y, which is part of every bag of $\mathbb{K}_Y$. Therefore, P_Y contains no vertex in $B_\ell \cup B_r$ (since $\beta(\mathbb{K}_Y) \cap (B_\ell \cup B_r) \subseteq C_Y$). In other words $P_Y \subseteq I_X$. Furthermore, as every vertex of P_Y is an internal vertex of $P \subseteq P(\mathbb{O})$, we have $P_Y \subseteq \mathsf{base}(\mathbb{O})$. Let $u \in B_i$ and $v \in B_j$ be the endpoints of P_Y. Now, let P'_Y be another induced path between u and v in $G[\beta(\mathbb{K}_Y) \setminus C_Y]$, and observe that $P'_Y \subseteq I_X$. Let us note that when $\mathbb{O}$ is a nice chordless cycle, we define the paths P, P_Y and P'_Y in a similar manner. We then have the following lemma.

Lemma 6.27. *There is a nice-obstruction $\mathbb{O}'$ which is not covered by $\mathcal{W}$ such that $\mathbb{O}' \subseteq (\mathbb{O} - P_Y) \cup P'_Y$.*

Proof. Since every vertex of P'_Y lies in I_X, the neighborhood of these vertices is contained in $\beta(\mathbb{K}_X) \cup M_A$. Recall that $P = \mathbb{O} \cap (\beta(\mathbb{K}_X) \setminus C_X)$ is an induced path from a vertex say, $x \in B_\ell$ to a vertex say, $y \in B_r$ that contains a vertex in $\beta(\mathbb{K}_X) \setminus (B_\ell \cup B_r)$. By x_o and y_o we denote the neighbors of x and y in P, respectively. Let P' be an induced path between x and y in $G[(V(P) \setminus V(P_Y)) \cup V(P'_Y)]$. Also, let $\mathbb{O}' = G[(V(\mathbb{O}) \setminus V(P)) \cup V(P')]$, and x_c and y_c be the neighbors of x and y in P', respectively. Notice that all the internal vertices of P' except possibly x_c and y_c are contained in I_X. Moreover, if $x_c \notin I_X$ then $x_c \in B_\ell$ and hence $x_c = x_o$, which follows from Observation 6.26 and the fact that P_Y, P'_Y do not contain any vertex from $B_\ell \cup B_r$. Similarly, if $y_c \notin I_X$ then $y_c \in B_r$ and $y_c = y_o$. Let $T = (N_{P'}[x] \cap B_\ell) \cup (N_{P'}[y] \cap B_r)$. The above discussion implies that $P_f = G[(V(P(\mathbb{O})) \setminus V(P)) \cup V(P')]$ is an induced path from t_ℓ to t_r. Here, t_ℓ and t_r are the base terminals of $\mathbb{O}$ if it is an AW, and otherwise $\mathbb{O}$ is a chordless cycle in which case t_ℓ and t_r are neighbors of x and y, respectively in $\mathbb{O}$ which are different from their neighbors in P. Notice that for each $w \in V(P') \setminus T$ we have $N_G(w) \subseteq \beta(\mathbb{K}_X) \cup M_A$. Therefore, from the above discussions, if $\mathbb{O}$ is a chordless cycle then $\mathbb{O}'$ is a chordless cycle on at least 4 vertices. Here, we rely on the fact that there are at least 10 vertices from M in $\mathbb{O}$ and they all lie in M_P. Next, consider the case when $\mathbb{O}$ is an AW. Notice that as P contains a vertex in $\mathbb{K}_Y - C_Y$, therefore the centers c_1, c_2 of $\mathbb{O}$ belong to $C_X \cup M_A$ (see Observation 6.16). This implies that each vertex in P' is adjacent to c_1 and c_2. Finally, recall that there are at least 4 vertices in $\mathbb{O} - \beta(\mathbb{K}_X)$, as M is a 9-redundant solution and $\mathbb{O}$ is not covered by $\mathcal{W}$. Now it follows that $\mathbb{O}' = (\mathbb{O} - P) \cup P'$ is also an obstruction. In each of the cases, by construction we have that $\mathbb{O}'$ is $\mathbb{K}_X$-nice and is not covered by $\mathcal{W}$. Moreover, $\mathbb{O} - \beta(\mathbb{K}_X) = \mathbb{O}' - \beta(\mathbb{K}_X)$. Therefore, it follows that $\mathbb{O}$ is a nice-obstruction which is not covered by $\mathcal{W}$. $\square$

In the following, by a **separator** in $\mathbb{K}_Y$, we mean a minimal separator of $B_i \setminus C_Y$ and $B_j \setminus C_Y$ in the graph $G[\beta(\mathbb{K}_Y) \setminus C_Y]$. Then we apply Lemma 6.27 to derive the fact that any minimal solution either does not intersect $\beta(\mathbb{K}_Y) \setminus C_Y$ or contains a separator in $\mathbb{K}_Y$. Furthermore, we can replace this separator with any other separator and the resulting set is also a solution. For a set $S \subseteq V(G)$, by S_Y we denote the set $S \cap (\beta(\mathbb{K}_Y) \setminus C_Y)$.

Lemma 6.28. *Let S be a solution of size at most $k+2$ in G that contains a vertex in $\beta(\mathbb{K}_Y) \setminus C_Y$. Then either S_Y is a separator in $\mathbb{K}_Y$, or else $S \setminus S_Y$ is also a solution. Furthermore, if S_Y is a separator then for any other separator S^*_Y in $\mathbb{K}_Y$ such that $S^* = (S \setminus S_Y) \cup S^*_Y$ has size at most $k + 2$, the set S^* is also a solution.*

Proof. First suppose that S_Y is not a separator in $\mathbb{K}_Y$. As S is a solution of size at most $k + 2$ in G, it hits all the sets in $\mathcal{W}$. By our assumptions, S does not separate $B_i \setminus C_Y$ and $B_j \setminus C_Y$ in $G[\beta(\mathbb{K}_Y) \setminus C_Y]$. Let $S' = S \setminus S_Y$. If S' is not a solution, there is an obstruction $\mathbb{O}'$ in the graph $G - S'$, and note that $\mathbb{O}'$ contains a vertex of $S \setminus S' \subseteq \beta(\mathbb{K}_Y) \setminus C_Y$. Since $M \cap S = M \cap S'$, it follows that S' also hits all the sets in $\mathcal{W}$. Hence the obstruction $\mathbb{O}'$ is not covered by $\mathcal{W}$, and $|\mathbb{O}' \cap M| \geq 10$. Therefore by Lemma 6.24, we obtain a nice-obstruction $\mathbb{O}$ in $G - S'$. Now we consider the obstruction $\mathbb{O}'$ in the graph G. Clearly, $\mathbb{O}$ also contains a vertex in $S \setminus S'$, and by Corollary 6.25, it follows that $V(\mathbb{O}) \cap G[\beta(\mathbb{K}_X) \setminus C_X]$ is a path P between a vertex of $B_\ell \setminus C_X$ and a vertex of $B_r \setminus C_X$ that is disjoint from C_X. Let $P_Y = P \cap \beta(\mathbb{K}_Y)$ and note that P_Y contains a vertex of $S \setminus S' \subseteq \beta(\mathbb{K}_Y) \setminus C_Y$. Let $u \in B_i$ and $v \in B_j$ be the endpoints P_Y. Since S_Y is not a separator in $\mathbb{K}_Y$, there is an induced path P'_Y between u and v in $G[\beta(\mathbb{K}_Y) \setminus C_Y]$ that is disjoint from S. Here, we rely on the fact that B_ℓ and B_r are cliques. Now, by Lemma 6.27, we have an obstruction $\mathbb{O}'' \subseteq G[(V(\mathbb{O}) \setminus V(P_Y)) \cup V(P'_Y)$ that is not covered by $\mathcal{W}$ in G, and by construction it is disjoint from S. But this is a contradiction. Therefore S' must also be a solution.

Now suppose that S_Y is a separator. We now argue that if S^*_Y is another separator, such that $S^* = (S \setminus S_Y) \cup S^*_Y$ has size at most $k + 2$, then S^* is also a solution. Suppose not, then we can argue that there is a nice-obstruction $\mathbb{O}$ in $G - S^*$, such that it contains a vertex of $S_Y \subseteq \beta(\mathbb{K}_Y) \setminus C_Y$. Then, as before, we obtain a path P_Y in $G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus C_Y)]$ between a vertex in $B_i \setminus C_Y$ and a vertex in $B_j \setminus C_Y$ in $\mathbb{K}_Y - C_Y$. But this contradicts the fact that S^*_{ij} is a separator. Therefore S^* must also be a solution. This concludes the proof of this lemma. $\square$

Corollary 6.29. *(i) If S is a minimal solution in G and $|S| \leq k+2$, then S_Y is either a minimal separator or $\emptyset$.*

(ii) If S is an optimum solution in G and $|S| \leq k+2$, then S_Y is either a minimum separator or $\emptyset$.

Let us select a minimum sized separator S^*_Y in $\mathbb{K}_Y$. Then we have the following lemma which follows from the proof of Lemma 6.28 and Corollary 6.29.

Lemma 6.30. *Let S be any minimal solution to the instance G of size at most $k+2$. Then there is a solution S' of cardinality at most $|S|$ such that $S' \cap (\beta(\mathbb{K}_Y) \setminus C_Y)$ is either the empty set or equal to S^*_Y.*

Let us recall the following fact about interval graphs and their clique path decomposition. In a clique path, any minimal separator is intersection of two adjacent bags [7]. Observe that, by definition $S^*_Y \cup C_Y$ is a separator in the clique path $\mathbb{K}$. We now have the following marking rule.

> **Marking Scheme VII.** For each pair of consecutive marked bags B_i, B_j in $\mathbb{K}_X$, let $\mathbb{K}_Y = \mathbb{K}[B_i, B_j]$. Then mark a pair of bags $B, B' \in \mathbb{K}_Y$ such that $B \cap B' = S^*_Y \cup C_Y$.

We note using the above marking scheme, we mark at most $\boxed{\mathcal{O}(\eta^{16}|M|k^3)}$ bags in $\mathbb{K}$, which follows from the number of bags marked by Marking Scheme VI. Now following Marking Scheme VII, we consider the problem of reducing the set of unmarked vertices in $\mathbb{K}_Y = \mathbb{K}[B_i, B_j]$, where B_i, B_j are two consecutive marked bags in a nice-clique path $\mathbb{K}_X$.

Lemma 6.31. *Let v be an unmarked vertex in $\mathbb{K}_Y$ such that v is contained in only one bag. Then (G, k) is a* Yes *instance of* IVD *if and only if $(G - \{v\}, k)$ is a* Yes *instance of* IVD.

Proof. In the forward direction, let S be a solution in G of size at most k. Clearly, S is a solution in $G - \{v\}$ as well. Now we consider the reverse direction. Let S be a solution of size at most k in $G - \{v\}$ and suppose that it is not a solution in G. Observe that $S \cup \{v\}$ is a solution in G of cardinality at most $k+1$, and therefore it hits each set in $\mathcal{W}$. Furthermore, as $v \notin M$, S hits every set in $\mathcal{W}$. Now consider an obstruction $\mathbb{O}$ in $G - S$, and clearly it includes v. It follows that the obstruction $\mathbb{O}$ is not covered by $\mathcal{W}$, and $V(\mathbb{O}) \cap M$ contains at least 10 vertices. Let us consider $\mathbb{O}$ in the graph G along with the set S. Observe that $N(v) \subseteq B \cup M_A$, where B is the (unique) bag in $\mathbb{K}_Y$ containing v. And therefore every pair of vertices in $(B \cup M_A) \setminus S$ is either an edge in $G - S$ or a pair in $\mathcal{W}$ (using Observation 6.13). Therefore v doesn't have a pair of non-adjacent neighbors in $\mathbb{O}$. Hence $\mathbb{O}$ is not a chordless cycle, and so it is an AW. Now, by Lemma 6.24, there is a nice-obstruction $\mathbb{O}'$ in $G - S$. And note that all terminals of $\mathbb{O}'$ lie in marked bags. If $v \in V(\mathbb{O}')$, then as v is an unmarked vertex, by Observation 6.16 and Corollary 6.25, v lies in $\mathsf{base}(\mathbb{O})$ and therefore $N(v)$ must contain a pair of non-adjacent vertices, which is a contradiction. But then v is not part of the obstruction $\mathbb{O}'$. This implies that that $\mathbb{O}'$ is an obstruction in $G - (S \cup \{v\})$, which is also a contradiction. Hence, S must also be a solution in G. This concludes the proof of this lemma. $\square$

The above lemma gives the following reduction rule.

Reduction Rule 6.2. *Let $\mathbb{K}_X$ be a nice-clique path, and let B_i, B_j are a pair of consecutive marked bags. Then pick a unmarked vertex in $\mathbb{K}_Y = \mathbb{K}[B_i, B_j]$ that is contained in only one bag, and delete it from the graph G. The resulting instance is $(G - \{v\}, k)$.*

If the above reduction rule is not applicable, then there are no unmarked vertices in any nice-clique path $\mathbb{K}_X$ that are contained in only one bag. Then observe that for any bag unmarked B in $\mathbb{K}_X$ we have $B = (B \cap B^{-1}) \cup (B \cap B^{+1})$. Let us now consider the remaining of the unmarked vertices in $\mathbb{K}_Y = \mathbb{K}[B_i, B_j]$, where B_i, B_j are a pair of consecutive marked bags in $\mathbb{K}_X$.

Lemma 6.32. *Let $\mathbb{K}_Y$ contain an unmarked vertex. Then there is an edge (u, v) such that at least one of its endpoints is an unmarked vertex, and there is only one bag in $\mathbb{K}_Y$ that contains this edge.*

Proof. Let us walk in $\mathbb{K}_Y$ starting from B_i, and let B be the first bag in $\mathbb{K}_Y$ that contains an unmarked vertex. Let us partition the bag B into three parts as follows, $A_2 = B^{-1} \cap B^{+1} \subseteq B$, $A_1 = (B \cap B^{-1}) \setminus A_2$ and $A_3 = B \cap B^{+1} \setminus A_2$. Note that, $B \cap B^{-1} = A_1 \cup A_2$, and $B \cap B^{+1} = A_2 \cap A_3$. Note that $A_1 \neq \emptyset$, otherwise $B = A_2 \cup A_3 \subseteq B^{+1}$ which is a contradiction as B is a maximal clique in the clique path $\mathbb{K}_X$, and hence $B \not\subseteq B^{+1}$. Therefore $A_1 \neq \emptyset$, and similarly $A_3 \neq \emptyset$. Now consider an unmarked vertex $u \in B$ and observe that $u \in A_3$, by choice of B. Next we choose a vertex $v \in A_1$ and clearly it is distinct from u. Furthermore, as $v \notin B^{+1}$ and $u \notin B^{-1}$, we have that the edge (u, v) is present only in B. ☐

In the following, we select an edge $e = (u, v)$ given by Lemma 6.32 that lies in $\mathbb{K}_Y = \mathbb{K}[B_i, B_j]$, for some pair of consecutive marked bags B_i, B_j in the nice-clique path $\mathbb{K}_X$. We call such an edge an **irrelevant edge**. Note that, by construction, $u, v \notin C_Y$ and therefore they belong to I_X.

Lemma 6.33. *Let (u, v) be an irrelevant edge in $\mathbb{K}_Y$. Then there is no minimal separator of $B_i \setminus C_Y$ and $B_j \setminus C_Y$ in $\mathbb{K}_Y - C_Y$ that contains both u and v.*

Proof. Recall that $\mathbb{K}_Y - C_Y$ is a clique path with endpoint bags $B_i - C_Y$ and $B_j - C_Y$. Therefore every minimal separator of these endpoint bags is the intersection of a pair of adjacent bags in $\mathbb{K}_Y - C_Y$. If both u and v were in a minimal-separator, then the edge (u, v) appears in at least two bags, which is a contradiction. Therefore, there is no minimal separator that contains both u and v. ☐

Observation 6.34. *A minimal solution of size at most $k + 2$ in G contains at most one of u and v, where (u, v) is an irrelevant edge.*

Proof. Let S be a minimal solution in G that contains both of u and v. Then, as S contains a vertex of $\beta(\mathbb{K}_Y) \setminus C_Y$, by Corollary 6.29 we have $S_Y = S \cap (\beta(\mathbb{K}_Y) \setminus C_Y)$ is a minimal separator. Now, by our assumptions, S_Y contains both u and v, whereas by Lemma 6.33, no minimal separator can contain both these vertices. This is a contradiction. ☐

Lemma 6.35. *Let $e = (u, v)$ be an irrelevant edge in $G[\beta(\mathbb{K}_Y)] - C_Y$, where u is an unmarked vertex. Then (G, k) is a* Yes *instance of* IVD *if and only if $(G/e, k)$ is a* Yes *instance of* IVD.

Proof. Let z^* denote the vertex obtained by contracting the irrelevant edge $e = (u, v)$. Let S be a solution of size k in G. Observe that, we can assume S is a minimal solution, and therefore it does not contain both u and v. Let $S' = (S \setminus \{u, v\}) \cup \{z^*\}$ whenever S contains at least one of u, v and $S' = S$ otherwise. In the first case, observe that $G/e - S'$ is isomorphic to $G - (S \cup \{u, v\})$. And in the second case $G/e - S'$ is isomorphic to $(G - S)/e$. As interval graphs are closed under edge-contractions and vertex deletions (Observation 6.1), we have that S' is a solution in G/e of size at most k.

Now suppose that S' is a solution of size at most k in G/e. We have two cases depending on whether or not $z^* \in S'$. First consider the case when $z^* \in S'$. Then $S = (S' \setminus \{z^*\}) \cup \{u, v\}$

is a solution of size $k+1$ in G, as $G - S$ is isomorphic to $G/e - S'$. Now, as S is a solution of size at most $k+1$, it must hit each set in $\mathcal{W}$. Moreover, $S \setminus \{u, v\}$ hits each set in $\mathcal{W}$, as $u, v \notin M$. Observe that S contains an unmarked vertex in $\mathbb{K}_Y$ (since $u, v \in \beta(\mathbb{K}_Y)$). Therefore, by Lemma 6.28, it follows that either there is a strict subset S'' of S that is also a solution, or $S_Y = S \cap (\beta(\mathbb{K}_Y) \setminus C_Y)$ is a separator of $B_i \setminus C_Y$ and $B_j \setminus C_Y$ in $G[\beta(\mathbb{K}_Y) - C_Y]$. In the first case, $S^* = S''$ is a solution in G of size at most k. In the second case, observe that S_Y cannot be a minimal separator, as that will contradict Lemma 6.33. Therefore, as $u, v \in S_Y$, there is a strict subset S'_Y that includes at most one of u an v, which is also separator. Then by Lemma 6.28, $S^* = (S \setminus S_Y) \cup S'_Y$ is also a solution in G, and note that it has size at most k. Hence we have obtained a solution S^* in G of size at most k.

Now consider the case when $z^* \notin S'$. In this case, let $S = S' \cup \{u, v\}$, and observe that it has size at most $k+2$. As $G - S$ is isomorphic to $G/e - (S' \cup \{z^*\})$, we have that S is a solution in G. As $\mathcal{W}$ is $(k+2)$-necessary, S hits each set in $\mathcal{W}$, which then implies that S' hits each set in $\mathcal{W}$. We claim that S' is a solution of size k in G. Suppose not and let there be an obstruction $\mathbb{O}'$ in $G - S'$. As S' hits $\mathcal{W}$, we have that $\mathbb{O}'$ is not covered by $\mathcal{W}$ and $V(\mathbb{O}') \cap M$ contains at least 10 vertices. Now from Lemma 6.24, there is a nice-obstruction $\mathbb{O}$ in $G - S'$ and note that is not covered by $\mathcal{W}$.

First suppose that $V(\mathbb{O}) \cap \{u, v\} = \emptyset$. Then clearly $\mathbb{O}$ is present in G/e, and furthermore it is disjoint from S'. This is a contradiction. Next, suppose that $V(\mathbb{O}) \cap \{u, v\}$ is one of u or v. We claim that $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$ contains an obstruction. Note that $\mathbb{O}$ is not covered by $\mathcal{W}$. As $u, v \in \beta(\mathbb{K}_Y) \setminus C_Y$, they lie in I_X. Therefore $N_G(u) \cup N_G(v) \subseteq \beta(\mathbb{K}_X) \cup M_A$, and hence u, v have no neighbors in $V(\mathbb{O}) \setminus (\beta(\mathbb{K}_X) \cup M_A)$. Now, as $P = \mathbb{O}[V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X)]$ contains a vertex from I_X, by Corollary 6.25, P must be an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$ such that $P - (B_\ell \cup B_r)$ is an induced path contained in I_X. Let us also note that P must contain at least 3 vertices. Now we have two following cases.

- Consider the case when $\mathbb{O}$ is a chordless cycle. As $\mathbb{O}$ is a nice-obstruction we have $|V(\mathbb{O}) \cap M| \geq 10$. And as P contains at least 3 vertices, $V(\mathbb{O}) \cap M_A = \emptyset$ (using Observation 6.13). Hence $(N_G(u) \cup N_G(v)) \cap (V(\mathbb{O}) \cap M) = \emptyset$. Now we can conclude that $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$ contains a chordless cycle, by considering a vertex $m \in V(\mathbb{O}) \cap M$ and the two induced paths between m and z^* in $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$.

- Next, we consider the case when $\mathbb{O}$ is an AW. As $\mathbb{O}$ is a nice-obstruction that contains a vertex from I_X, it follows that $P \subseteq P(\mathbb{O})$ and $P \cap I_X \subseteq \mathsf{base}(\mathbb{O})$ (see the proof of Lemma 6.24, Lemma 6.27 and Lemma 6.20). Note that $\{u, v\} \cap V(P) \subset V(\mathsf{base}(\mathbb{O}))$. Furthermore, $V(P(\mathbb{O})) \cap M_A = \emptyset$, as P contains at least 3 vertices and any vertex in M_A is adjacent to every vertex in $\mathbb{K}_X$ (using Observation 6.13). Also note that the shallow terminal t lies outside $\mathbb{K}_X$, as P contains a vertex from every bag in $\mathbb{K}_X$ (using Corollary 6.25). Hence, $V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \cup M_A) = V(P) \cup \{c_1, c_2\}$. Therefore $(N_G(u) \cup N_G(v)) \cap V(\mathbb{O}) \subseteq V(P) \cup \{c_1, c_2\}$. Furthermore $\{c_1, c_2\} \subseteq C_X \cup M_A$ (see the proof of Lemma 6.24). And since $|\mathsf{base}(\mathbb{O}) \cap M| \geq 5$, we have that P is a strict subset of $P(\mathbb{O})$. Therefore, $(N_G(u) \cup N_G(v)) \cap (V(\mathbb{O}) \setminus \{c_1, c_2\})$ is a strict subset of $V(P(\mathbb{O}))$ and $u, v \in N_G(c_1) \cap N_G(c_2)$. Hence $G/e[(V(P(\mathbb{O})) \setminus \{u, v\}) \cup \{z^*\}]$ contains an induced path from t_ℓ to t_r with at least 6 internal vertices, where t_ℓ and t_r are base terminals of $\mathbb{O}$. Now it follows that $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$ contains an AW of the same type as $\mathbb{O}$. Further observe that this obstruction lies in $G/e - S'$, which is a contradiction.

Now we consider the case that both u, v are present in $\mathbb{O}$. We claim that $\mathbb{O}/e$ is an obstruction in G/e. Indeed, if $\mathbb{O}$ is a chordless cycle, then as it contains at least 10 vertices in M, it follows that $\mathbb{O}/e$ is also a chordless cycle on at least 9 vertices. Otherwise, $\mathbb{O}$ is a nice AW. Now, recall that u is an unmarked vertex in $\beta(\mathbb{K}_Y) \subseteq \beta(\mathbb{K}_Y)$ and observe that the vertex u lies in I_X. Let

$P = \mathbb{O}[V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus C_X)$ and observe that $P \cap I_X \neq \emptyset$. Therefore, by Corollary 6.25, we have that P is an induced path between a vertex in $B_\ell \setminus C_X$ and a vertex in $B_r \setminus C_X$, such that $P - (B_\ell \cup B_r)$ is an induced path contained in $G[I_X]$. Let $P_Y = P[V(P) \cap \beta(\mathbb{K}_Y)]$ and observe that it contains the edge (u, v). Furthermore, $V(P_Y) \cap (B_\ell \cup B_r) = \emptyset$ by the construction of $\mathbb{K}_Y$ (from the marking schemes). Therefore all vertices in P_Y, and in particular the vertices u and v, must be internal vertices of $P(\mathbb{O})$, i.e. they are in $\mathsf{base}(\mathbb{O})$ (see the proof of Lemma 6.24). Finally, recall that $\mathsf{base}(\mathbb{O})$ contains at least 5 vertices of M. Therefore, $P(\mathbb{O})/e$ is an induced path on at least 6 vertices between t_ℓ and t_r. Hence, it follows that $\mathbb{O}/e$ is an AW of the same type as $\mathbb{O}$, and further it is present in G/e. Finally observe that $\mathbb{O}/e$ is an obstruction in G/e that is disjoint from S'. This is a contradiction.

Having obtained a contradiction in all cases, we must conclude that S' is a solution in G, and recall that it has size at most k. This concludes the proof of this lemma. □

The above lemma (Lemma 6.35) gives us the following reduction rule.

Reduction Rule 6.3. *Let (u, v) be an irrelevant edge in $G[\beta(\mathbb{K}_Y)] - C_Y$ where u is an unmarked vertex. Then contract the edge (u, v) in the graph G. The resulting instance is $(G/e, k)$.*

By Reduction Rule 6.3, we may assume that there are no unmarked vertices in $\beta(\mathbb{K}_Y)$. Then applying this reduction rule over all pairs of consecutive marked bags in every nice-clique path, we conclude that all vertices in the clique path $\mathbb{K}$ are marked. Finally, we apply the above marking schemes and reduction rules for every clique path in $G - M$, and conclude that all the vertices in $G - M$ are marked. We now proceed to bounding the number of vertices in the graph.

7 Bounding the Number of Vertices

Let (G, k) be an instance of IVD on which none of the reduction rules apply. In the following we bound the number of vertices in G. Recall that we start by computing a 9-redundant solution M, whose size is bounded by $\mathcal{O}(k^{10})$ (see Lemma 3.1). Next, we consider the connected components of $G - M$. First, we bound the total number of vertices in the module components of $G - M$ by $\mathcal{O}(k^3|M|^6) = \mathcal{O}(k^{63})$ (see Observation 4.2). Then, we bound the total number of vertices in the non-module components of $G - M$ by a collection marking rules (and the non-applicability of a number of reduction rules). From Observation 4.2 we obtain that the number of non-module components in $G - M$ is bounded by $\mathcal{O}(k|M|) = \mathcal{O}(k^{11})$. We note that each non-module component is a clique path. Then, we consider a clique path $\mathbb{K}$ of a non-module connected component in $G - M$, and bound the size of the maximum clique in it by $\eta = \mathcal{O}(k|M|^{10}) = \mathcal{O}(k^{101})$ (see Lemma 5.4). Next, we focus on bounding the number of bags in a clique path $\mathbb{K}$ that is a non-module component in $G - M$. In the following, for a fixed (non-module) clique path $\mathbb{K}$, we summarize the number of bags we mark using each of our bag-marking schemes in Section 6.

1. Using Marking Scheme I, we mark at most $\mathcal{O}(\eta|M|)$ bags in $\mathbb{K}$.
2. Using Marking Scheme II, we mark at most $\mathcal{O}(k^2\eta^7|M|)$ bags in $\mathbb{K}$.
3. Using Marking Scheme III, we mark at most $\mathcal{O}(\eta^3|M|)$ bags in $\mathbb{K}$.
4. Using Marking Scheme IV, we mark at most $\mathcal{O}(\eta^3|M|)$ bags in $\mathbb{K}$.
5. Using Marking Scheme V, we mark at most $\mathcal{O}(k^3\eta^{15}|M|)$ bags in $\mathbb{K}$.
6. Using Marking Scheme VI, we mark at most $\mathcal{O}(k^3\eta^{16}|M|)$ bags in $\mathbb{K}$.
7. Using Marking Scheme VII, we mark at most $\mathcal{O}(k^3\eta^{16}|M|)$ bags in $\mathbb{K}$.

From the above, we obtain that the number of marked bags for each (non-module) clique path is upper bounded by $\mathcal{O}(k^3\eta^{16}|M|) = \mathcal{O}(k^{1629})$. Further, since none of the reduction rules is applicable, there is no vertex in G that belongs to an unmarked bag of a non-module component. Moreover, there are at most $\mathcal{O}(k^{11})$ non-module components in $G - M$, and a bag in a clique path of a non-module component has size at most η. Thus, the total number of vertices in G is bounded by $\mathcal{O}(k^{1629} \cdot k^{11} \cdot k^{101}) = \mathcal{O}(k^{1741})$.

8 Conclusion

In this paper, we proved that the IVD problem admits a polynomial kernel. We remark that the degree in the polynomial that bounds the kernel size can be improved to be about a 100 at the cost of significantly more involved arguments. In particular, this can be done by considering a solution M of lower redundancy and far more involved case analysis for bounding the clique size and clique paths of $G - M$ in Sections 5 and 6. However, obtaining a kernel of size around $O(k^{10})$ will require new ideas. We leave this as an interesting open problem. We also believe that our techniques and methods, especially the Two Families Lemma (Lemma 1.1), will be useful in other algorithmic applications.

References

[1] A. Agrawal, D. Lokshtanov, P. Misra, S. Saurabh, and M. Zehavi, *Feedback vertex set inspired kernel for chordal vertex deletion*, in Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms, (SODA), 2017, pp. 1383–1398. 4, 39

[2] ——, *Feedback vertex set inspired kernel for chordal vertex deletion*, CoRR, abs/1707.04917 (2017). 4

[3] A. Agrawal, D. Lokshtanov, A. E. Mouawad, and S. Saurabh, *Simultaneous Feedback Vertex Set: A Parameterized Perspective*, in STACS 2016, vol. 47 of LIPIcs, 2016, pp. 7:1–7:15. 4

[4] A. Bar-Noy, R. Bar-Yehuda, A. Freund, J. Naor, and B. Schieber, *A unified approach to approximating resource allocation and scheduling*, J. ACM, 48 (2001), pp. 1069–1090. 1

[5] H. L. Bodlaender, R. G. Downey, M. R. Fellows, and D. Hermelin, *On problems without polynomial kernels*, Journal of Computer and System Sciences, 75 (2009), pp. 423–434. 1

[6] B. Bollobás, *On generalized graphs*, Acta Mathematica Hungarica, 16 (1965), pp. 447–452. 3

[7] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph classes: a survey*, SIAM, 1999. 1, 39, 58

[8] Y. Cao, *Unit interval editing is fixed-parameter tractable*, in ICALP, vol. 9134 of Lecture Notes in Computer Science, Springer, 2015, pp. 306–317. 4

[9] ——, *Linear recognition of almost interval graphs*, in SODA, 2016, 2016, pp. 1096–1115. 1, 4, 8

[10] Y. Cao and D. Marx, *Interval deletion is fixed-parameter tractable*, ACM Trans. Algorithms, 11 (2015), pp. 21:1–21:35. 1, 4, 13

[11] ——, *Chordal editing is fixed-parameter tractable*, Algorithmica, 75 (2016), pp. 118–137. 4

[12] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, *Parameterized Algorithms*, Springer, 2015. 1, 6

[13] M. Cygan, Ł. Kowalik, and M. Pilipczuk, *Open problems from workshop on kernels*, Available at `http://worker2013.mimuw.edu.pl/slides/worker-opl.pdf`, (2013). 1

[14] H. Dell and D. Marx, *Kernelization of packing problems*, in Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms, SODA, 2012, pp. 68–81. 1

[15] H. Dell and D. van Melkebeek, *Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses*, Journal of the ACM, 61 (2014), pp. 23:1–23:27. 1

[16] R. Diestel, *Graph Theory*, Springer-Verlag, Heidelberg, 4th ed., 2010. 5

[17] R. G. Downey and M. R. Fellows, *Fundamentals of Parameterized Complexity*, Texts in Computer Science, Springer, 2013. 1, 6

[18] A. Drucker, *New limits to classical and quantum instance compression*, SIAM Journal on Computing, 44 (2015), pp. 1443–1479. 1

[19] J. Flum and M. Grohe, *Parameterized Complexity Theory*, Texts in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, 2006. 1, 6

[20] F. V. Fomin, B. M. P. Jansen, and M. Pilipczuk, *Preprocessing subgraph and minor problems: When does a small vertex cover help?*, J. Comput. Syst. Sci., 80 (2014), pp. 468–495. 3

[21] F. V. Fomin, D. Lokshtanov, N. Misra, and S. Saurabh, *Planar F-deletion: Approximation, kernelization and optimal FPT algorithms*, in FOCS, 2012. 1, 4

[22] F. V. Fomin, D. Lokshtanov, F. Panolan, and S. Saurabh, *Efficient computation of representative families with applications in parameterized and exact algorithms*, Journal of the ACM, 63 (2016), pp. 29:1–29:60. 3, 7

[23] F. V. Fomin and S. Saurabh, *Kernelization methods for fixed-parameter tractability*, in Tractability, Cambridge Univ. Press, Cambridge, 2014, pp. 260–282. 1

[24] L. Fortnow and R. Santhanam, *Infeasibility of instance compression and succinct PCPs for NP*, Journal of Computer and System Sciences, 77 (2011), pp. 91–106. 1

[25] T. Fujito, *A unified approximation algorithm for node-deletion problems*, Discrete Appl. Math., 86 (1998), pp. 213–231. 1

[26] D. Fulkerson and O. Gross, *Incidence matrices and interval graphs*, Pacific journal of mathematics, 15 (1965), pp. 835–855. 6

[27] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980. 1, 6, 39, 40, 41

[28] J. Guo and R. Niedermeier, *Invitation to data reduction and problem kernelization*, SIGACT News, 38 (2007), pp. 31–45. 1

[29] D. Hermelin, S. Kratsch, K. Soltys, M. Wahlström, and X. Wu, *A completeness theory for polynomial (turing) kernelization*, Algorithmica, 71 (2015), pp. 702–730. 1

[30] D. Hermelin and X. Wu, *Weak compositions and their applications to polynomial lower bounds for kernelization*, in Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms, SODA, 2012, pp. 104–113. 1

[31] B. M. Jansen, *The power of data reduction: Kernels for Fundamental Graph Problems*, PhD thesis, Utrecht University, 2013. 1

[32] B. M. P. Jansen, D. Lokshtanov, and S. Saurabh, *A near-optimal planarization algorithm*, in SODA, 2014, pp. 1802–1811. 4

[33] B. M. P. Jansen and M. Pilipczuk, *Approximation and kernelization for chordal vertex deletion*, CoRR, abs/1605.03001 (2016). 1

[34] ——, *Approximation and kernelization for chordal vertex deletion*, in Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, 2017, pp. 1399–1418. 4, 39

[35] Y. Ke, Y. Cao, X. Ouyang, W. Li, and J. Wang, *Unit interval vertex deletion: Fewer vertices are relevant*, J. Comput. Syst. Sci., 95 (2018), pp. 109–121. 4

[36] D. Kendall, *Incidence matrices, interval graphs and seriation in archeology*, Pacific Journal of mathematics, 28 (1969), pp. 565–570. 1

[37] E. J. Kim and O. Kwon, *A polynomial kernel for block graph deletion*, in IPEC, vol. 43 of LIPIcs, 2015, pp. 270–281. 4

[38] E. J. Kim, A. Langer, C. Paul, F. Reidl, P. Rossmanith, I. Sau, and S. Sikdar, *Linear kernels and single-exponential algorithms via protrusion decompositions*, ACM Trans. Algorithms, 12 (2016), pp. 21:1–21:41. 4

[39] S. Kratsch, *Recent developments in kernelization: A survey*, Bulletin of the EATCS, 113 (2014). 1, 6

[40] S. Kratsch and M. Wahlstrom, *Representative sets and irrelevant vertices: New tools for kernelization*, 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, 0 (2012), pp. 450–459. 3

[41] D. C. Lay, *Linear algebra and its applications*, Linear Algebra and Its Applications, Pearson/Addison-Wesley, 2006. 7

[42] C. Lekkeikerker and J. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fundamenta Mathematicae, 51 (1962), pp. 45–64. 2

[43] J. M. Lewis and M. Yannakakis, *The node-deletion problem for hereditary properties is NP-complete*, J. Comput. Syst. Sci., 20 (1980), pp. 219–230. 1

[44] D. Lokshtanov, N. Misra, and S. Saurabh, *Kernelization - preprocessing with a guarantee*, in The Multivariate Algorithmic Revolution and Beyond - Essays Dedicated to Michael R. Fellows on the Occasion of His 60th Birthday, 2012, pp. 129–161. 1, 6

[45] L. Lovász, *A characterization of perfect graphs*, Journal of Combinatorial Theory, Series B, 13 (1972), pp. 95–98. 41

[46] C. Lund and M. Yannakakis, *On the hardness of approximating minimization problems*, J. ACM, 41 (1994), pp. 960–981. 1

[47] D. Marx, *A parameterized view on matroid optimization problems*, Theor. Comput. Sci., 410 (2009), pp. 4471–4479. 3

[48] D. Marx, *Chordal deletion is fixed-parameter tractable*, Algorithmica, 57 (2010), pp. 747–768. 4

[49] R. Niedermeier, *Invitation to fixed-parameter algorithms*, vol. 31 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2006. 1, 6

[50] J. G. Oxley, *Matroid theory*, vol. 3, Oxford University Press, USA, 2006. 7, 50

[51] M. Yannakakis, *Node-and edge-deletion NP-complete problems*, in STOC, New York, NY, USA, 1978, ACM, pp. 253–264. 1