Fine-grained Complexity of Rainbow Coloring and its Variants

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Abstract

Consider a graph G and an edge coloring $c_R : E(G) \to [k]$. A rainbow path between $u, v \in V(G)$ is a path P from u to v such that for all $e, e' \in E(P)$, where $e \neq e'$ we have $c_R(e) \neq c_R(e')$. The problem RAINBOW k-COLORING takes as an input a graph G, and the objective is to decide if there exists $c_R : E(G) \to [k]$ such that for all $u, v \in V(G)$ there is a rainbow path between u and v in G. Several variants of the RAINBOW k-COLORING have been studied. Two such variants are as follows. The SUBSET RAINBOW k-COLORING takes as an input a graph G and a set $S \subseteq V(G) \times V(G)$, and the objective is to decide if there exists $c_R : E(G) \to [k]$ such that for all $(u, v) \in S$ there is a rainbow path between u and v in G. The problem STEINER RAINBOW k-COLORING takes as an input a graph G and a set $S \subseteq V(G)$, and the objective is to decide if there exists $c_R : E(G) \to [k]$ such that for all $u, v \in S$ there is a rainbow path between u and v in G. In an attempt to resolve the open problems posed by Kowalik et al. (ESA 2016), in this paper we obtain the following results.

- For every $k \geq 3$, RAINBOW k-COLORING does not admit an algorithm running in time $2^{o(|E(G)|)} n^{\mathcal{O}(1)}$, unless ETH fails.
- For every $k \geq 3$, STEINER RAINBOW k-COLORING does not admit an algorithm running in time $2^{o(|S|^2)} n^{\mathcal{O}(1)}$, unless ETH fails.
- SUBSET RAINBOW k-COLORING admits an algorithm running in time $2^{\mathcal{O}(|S|)} n^{\mathcal{O}(1)}$. This also implies an algorithm running in time $2^{o(|S|^2)} n^{\mathcal{O}(1)}$ for STEINER RAINBOW k-COLORING, which matches the lower bound we obtained.

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1 Introduction

Graph connectivity is one of the fundamental properties in graph theory. Several connectivity measures like k-vertex connectivity, k-edge connectivity, hamiltonicity, etc. have been studied for graphs. Inspired by applications in secure data transfer, Chartrand et al. [8] defined an interesting connectivity measure, called rainbow connectivity, which is defined as follows. Let G be a graph and $c_R : E(G) \to [k]$ be an edge coloring of G. A rainbow path between $u, v \in V(G)$ is a path P from u to v such that for all $e, e' \in E(P)$, where $e \neq e'$ we have $c_R(e) \neq c_R(e')$. A graph with an edge coloring is said to be rainbow connected if for every pair of vertices there is a rainbow path between them. The problem RAINBOW k-COLORING takes as an input a graph G, and the goal is to decide whether there exists an edge coloring $c_R : E(G) \to [k]$ such that for all $u, v \in V(G)$, there is a rainbow path between u and v in G. The problem has received lot of attention recently, both from graph theoretic and algorithmic point of view, the details of which can be found, for instance in [9, 25, 26].

The problem Rainbow k-Coloring is a notoriously hard problem. It was conjectured by Caro et al. [4] that Rainbow k-Coloring is NP-hard even for $k = 2$. Chakraborty et al. [5] confirmed this conjecture by showing that the problem is NP-hard for $k = 2$. Later, Ananth et al. [3] showed that for every $k \geq 2$, Rainbow k-Coloring is NP-hard. An alternate proof was also given by Le and Tuza [23]. For the complexity of the problem on restricted graph classes see [5, 6, 7, 8]. The problem has received

Impagliazzo et al. introduced the Exponential time hypothesis (ETH) [18], which is used as a basis for proving qualitative lower bounds for computational problems. ETH states that the problem 3-SAT does not admit an algorithm running in time $2^{o(n)}n^{\mathcal{O}(1)}$, where n is the number of variables in the input 3-CNF formula. Since then it has been used to prove that several of the NP-hard problems like Independent Set, Hitting set, Chromatic Number, do not admit an algorithm running in sub-exponential time, assuming ETH (see the survey [27]).

Kowalik et al. [22] studied the fine-grained complexity of Rainbow k-Coloring and its variants. They showed that Rainbow k-Coloring neither admit an algorithm running in time $2^{o(|V(G)|^{3/2})}|V(G)|^{\mathcal{O}(1)}$, nor an algorithm running in time $2^{o(|E(G)|/\log|E(G)|)}|V(G)|^{\mathcal{O}(1)}$, unless ETH fails. They also studied a variant of Rainbow k-Coloring, called Subset Rainbow k-Coloring (to be defined shortly), which was introduced by Chakraborty et al. [5]. They showed that Subset Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}|V(G)|^{\mathcal{O}(1)}$ assuming ETH. Also, they designed an FPT algorithm for Subset Rainbow k-Coloring running in time $|S|^{\mathcal{O}(|S|)}n^{\mathcal{O}(1)}$, where S is a part of the input. For $k = 2$ they designed an algorithm running in time $2^{\mathcal{O}(|S|)}n^{\mathcal{O}(1)}$. They proposed yet another (parametric) variant of Rainbow k-Coloring, which they called Steiner Rainbow k-Coloring. Their lower bound result for Rainbow k-Coloring implies that Steiner Rainbow k-Coloring does not admit an algorithm running in time $2^{o|S|^{3/2}}n^{\mathcal{O}(1)}$. Moreover, their algorithm for Subset Rainbow k-Coloring gives an algorithm for Steiner Rainbow k-Coloring running in time $2^{\mathcal{O}(|S|^2\log|S|)}n^{\mathcal{O}(1)}$.

Our results. In this paper, we attempt to tighten the gaps in the fine-grained complexity of Rainbow k-Coloring, Subset Rainbow k-Coloring, and Steiner Rainbow k-Coloring, which was initiated by Kowalik et al. [22]. We now move to the description of each of our results.

The first problem that we study is Steiner Rainbow k-Coloring, which is formally defined below.

Steiner Rainbow k-Coloring	**Parameter:** |S|
Input: A graph G and a subset $S \subseteq V(G)$.	
Question: Does there exist an edge coloring $c_R : E(G) \to [k]$ such that for every $u, v \in S$, there is a rainbow path between u and v in G?	

In Section 3, we show that Steiner Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|S|^2)}n^{\mathcal{O}(1)}$ for every $k \geq 3$, thus resolving one of the problems posed by Kowalik et al. [22]. We give a reduction from k-Coloring on graphs of maximum degree $2(k-1)$ which does not admit an algorithm running in time $2^{o(n)}n^{\mathcal{O}(1)}$, assuming ETH. Our reduction starts by computing a harmonious coloring of the (bounded degree) input instance of k-Coloring, which forms an essential step in the construction of S for the instance of Steiner Rainbow k-Coloring that we create. The idea of using harmonious coloring for proving lower bound of the form $2^{o(\ell^2)}n^{\mathcal{O}(1)}$ was used by Agrawal et al. [1] to prove a lower bound for Split Contraction, when parameterized by the vertex cover number of the input graph. Here, ℓ is some parameter of the input instance. Also, the idea of partitioning vertices of the input graph based on some coloring scheme was used by Cygan et al. [10] to prove ETH based lower bounds for Graph Homomorphism and Subgraph Isomorphism.

The next problem we study is Rainbow k-Coloring, which is formally defined below.

Rainbow k-Coloring
Input: A graph G.
Question: Does there exist an edge coloring $c_R : E(G) \to [k]$ such that for every $u, v \in V(G)$, there is a rainbow path between u and v in G?

Kowalik et al. [22] conjectured that for every $k \geq 2$, Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, unless ETH fails. In Section 4, we resolve this conjecture for every $k \geq 3$. We give a reduction from k-Coloring on bounded degree graphs. Although, the general scheme of reduction is same as that for Steiner Rainbow k-Coloring, but for this case the reduction is more involved. Furthermore, we require to distinguish between the cases for k being odd and even in the gadget construction. Also, for the sake of reducing the complexity of gadget construction, we separate the case for $k = 3$ and $k > 3$.

Finally, we study the problem Subset Rainbow k-Coloring, which is formally defined below.

Subset Rainbow k-Coloring **Parameter:** |S|
Input: A graph G and a subset $S \subseteq V(G) \times V(G)$.
Output: An edge coloring $c_R : E(G) \to [k]$ such that for every $(u, v) \in S$, there is a rainbow path between u and v in G, if it exists. Otherwise, return *no*.

In Section 5 we design an FPT algorithm running in time $2^{\mathcal{O}(|S|)}n^{\mathcal{O}(1)}$ for Subset Rainbow k-Coloring, for every fixed k. This resolves the conjecture of Kowalik et al. [22] regarding the existence of an algorithm running in time $2^{\mathcal{O}(|S|)}n^{\mathcal{O}(1)}$ for Subset Rainbow k-Coloring, and is an improvement over their algorithm, which runs in time $|S|^{\mathcal{O}(|S|)}n^{\mathcal{O}(1)}$, for $k \geq 3$. Our algorithm is based on the technique of color coding, which was introduced by Alon et al. [2]. Observe that Steiner Rainbow k-Coloring is a special case of Subset Rainbow k-Coloring. Hence, as a corollary we obtain an algorithm running in time $2^{\mathcal{O}(|S|^2)}n^{\mathcal{O}(1)}$ for Steiner Rainbow k-Coloring, which matches the lower bound we proved in Section 3.

2 Preliminaries

In this section, we state some basic definitions and introduce terminology from graph theory and algorithms. We also establish some of the notations that will be used throughout.

We denote the set of natural numbers by $\mathbb{N}$. For $k \in \mathbb{N}$, by $[k]$ we denote the set $\{1, 2, \cdots, k\}$. Let $f : X \to Y$ be a function. For $y \in Y$, by $f^{-1}(y)$ we denote the set $\{x \in X \mid f(x) = y\}$. For $X' \subseteq X$, by $f|_{X'}$ we denote the function $f|_{X'} : X' \to Y$ such that $f|_{X'}(x) = f(x)$, for all $x \in X'$. For an ordered set $R = X \times Y$, a function $f : R \to Z$, and an element $r = (x, y) \in R$, we slightly abuse the notation to denote $f(r) = f((x, y))$ by $f(x, y)$.

We use standard terminology from the book of Diestel [13] for the graph related terminologies which are not explicitly defined here. We consider finite simple graphs. For a graph G, by $V(G)$ and $E(G)$ we denote the vertex and edge sets of the graph G, respectively. For $v \in V(G)$, by $N_G(v)$ we denote the set $\{u \in V(G) \mid (v, u) \in E(G)\}$. We drop the subscript G from $N_G(v)$ when the context is clear. For $C, C' \subseteq V(G)$, we say that there is an edge between C and C' in G if there exists $u \in C$ and $v \in C'$ such that $(u, v) \in E(G)$. A *path* $P = (v_1, v_2, \cdots, v_\ell)$ is a graph with vertex set as $\{v_1, v_2, \cdots, v_\ell\}$ and edge set as

$\{(v_i, v_{i+1}) \mid i \in [l-1]\}$. Furthermore, we call such a path as a path between v_1 and v_ℓ. The length of a path is the number of edges in it.

A vertex *coloring* of a graph G with $k \in \mathbb{N}$ colors is a function $\varphi : V(G) \to [k]$. For such a vertex-coloring, we will call the sets $C_1, C_2, \cdots C_k$ as color classes, where $C_i = \{v \in V(G) \mid \varphi(v) = i\}$ for $i \in [k]$. A vertex-coloring φ of G is said to be *proper* if for each $(u, v) \in E(G)$, $\varphi(u) \neq \varphi(v)$. In this paper, by vertex coloring, we will always refer to a proper vertex coloring. A *harmonious coloring* of a graph G is a vertex-coloring $\varphi : V(G) \to [k]$, with color classes $C_1, C_2, \cdots, C_k$ such that for all $i, j \in [k]$, where $i \neq j$ there is at most one edge between C_i and C_j in G. An edge coloring of a graph G with k colors is a function $\phi : E(G) \to [k]$. A path P in G is said to be a *rainbow path* if for all $e, e' \in E(P)$ with $e \neq e'$ we have $\phi(e) \neq \phi(e')$. An edge coloring is said to be a *rainbow coloring* if for all $u, v \in V(G)$ there is a $u - v$ rainbow path in G. We drop the prefix vertex and edge from vertex coloring and edge coloring whenever the context is clear. For a graph G with an edge coloring $c : E(G) \to [k]$, and a path $P = (v_1, v_2, \cdots, v_{\ell-1}, v_\ell)$ in G by $(v_1 \overset{c_1}{—} v_2 \overset{c_2}{—} \cdots \overset{c_{\ell-2}}{—} v_{\ell-1} \overset{c_{\ell-1}}{—} v_\ell)$ we denote the path P annotated with the color of its edges. Here, $c(v_i, v_{i+1}) = c_i$, for $i \in [\ell - 1]$.

Parameterized complexity. A parameterized problem Π is a subset of $\Gamma^* \times \mathbb{N}$, where Γ is a finite alphabet set. An instance of a parameterized problem is a tuple (x, κ), where κ is called the *parameter*. A parameterized problem is said to be *fixed-parameter tractable* (FPT) if, for a given instance (x, κ), we can decide $(x, \kappa) \in \Pi$ in time $f(\kappa) \cdot |x|^{\mathcal{O}(1)}$, where $f(\cdot)$ is a computable function depending only on κ. For more details on parameterized complexity we refer to the books of Downey and Fellows [14], Flum and Grohe [16], Niedermeier [30], and the recent book by Cygan et al. [12].

3 Lower bound for Steiner Rainbow k-Coloring

In this section, we show that for every $k \geq 3$, Steiner Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|S|^2)} n^{\mathcal{O}(1)}$, unless ETH fails. Towards this we give an appropriate reduction from k-Coloring on graphs of maximum degree $2(k-1)$. We note that k-Coloring does not admit an algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ unless ETH fails [19]. Moreover, assuming ETH, 3-Coloring does not admit an algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ on graph of maximum degree 4 [20, 11]. This follows from the fact that 3-Coloring does not admit such an algorithm, and a reduction from an instance G of 3-Coloring to an equivalent instance G' of 3-Coloring, where G' is a graph with maximum degree 4 with $|V(G')| \in \mathcal{O}(|V(G)|)$ (see Theorem 4.1 [17]). In fact, we can show that k-Coloring does not admit an algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ on graph of maximum degree $2(k-1)$ (folklore). This result can be obtained (inductively) by giving a reduction from an instance G of $(k-1)$-Coloring on graphs of degree at most $2(k-2)$ to an instance of k-Coloring on a graphs of bounded average degree (by adding global vertex), and then using an approach similar to that in Theorem 4.1 in [17] we can obtain an (equivalent) instance of k-Coloring where the degree of the graph is bounded by $2(k-1)$.

Given an instance G of k-Coloring on n vertices and degree bounded by $2(k-1)$, we start by computing a harmonious coloring φ of G with $t \in \mathcal{O}(\sqrt{n})$ color classes such that each color class contains at most $\mathcal{O}(\sqrt{n})$ vertices. A harmonious coloring can be computed in polynomial time on bounded degree graphs using $\mathcal{O}(\sqrt{n})$ colors with each color class having at most $\mathcal{O}(\sqrt{n})$ vertices [11, 15, 24, 28]. Let $C_1, C_2, \cdots, C_t$ be the color classes of φ. Recall that for $i, j \in [t]$ with $i \neq j$ there is at most one edge between C_i and C_j in G. Moreover, C_i is an independent set in G, where $i \in [t]$. We create an instance G' of k-Coloring which has a harmonious coloring φ' with color classes $C'_1, C'_2, \cdots, C'_t$ such that for all $i, j \in [t]$, $i \neq j$ we have exactly one edge between C_i and C_j. Initially, we have $G = G'$ and $C'_i = C_i$, for all $i \in [t]$. For each $i, j \in [t]$, $i \neq j$ such that there is no edge between C_i and C_j in G we add two new vertices a_{ij} and a_{ji} to $V(G')$ and add the edge (a_{ij}, a_{ji}) to $E(G')$. Furthermore, we add a_{ij} to C'_i and a_{ji} to C'_{ji}. Observe that $|V(G')| \in \mathcal{O}(n)$, $|E(G')| \in \mathcal{O}(n)$, and for each $i \in [t]$, $|C'_i| \in \mathcal{O}(\sqrt{n})$. Also, for each $i, j \in [t]$, $i \neq j$ there is exactly one edge between C'_i and C'_j in G'. It is easy to see that G is a *yes* instance of k-Coloring if and only if G' is a *yes* instance of k-Coloring.

Hereafter, we will be working with the instance G' of k-Coloring, together with its harmonious coloring φ' with color classes $C'_1, C'_2, \cdots, C'_t$. Moreover, for $i, j \in [t]$, $i \neq j$ there is exactly one edge between C'_i and C'_j in G'.

We now move to the description of creating an equivalent instance $(\tilde{G}, S)$ of Steiner Rainbow k-Coloring, where $k \geq 3$. Initially, we have $V(\tilde{G}) = V(G')$. For $(u, v) \in E(G')$ we add $k - 3$ new vertices $x_1^{uv}, x_2^{uv}, \cdots, x_{k-3}^{uv}$ to $\tilde{G}$ and add all the edges in the path $(u, x_1^{uv}, \cdots, x_{k-3}^{uv}, v)$ to $E(\tilde{G})$. Note that for $k = 3$ we do not any new vertex and directly add the edge (u, v) to $\tilde{G}$. For each $i \in [t]$ we add a vertex c_i to $\tilde{G}$ and add all the edges in $\{(c_i, v) \mid v \in C'_i\}$ to $E(\tilde{G})$. Finally, we set $S = \{c_i \mid i \in [t]\}$. Notice that $|S| \in \mathcal{O}(\sqrt{n})$. In the following lemma we establish that G' is a *yes* instance of k-Coloring if and only if $(\tilde{G}, S)$ is a *yes* instance of Steiner Rainbow k-Coloring.

▶ **Lemma 1.** *G' is a* yes *instance of k-*Coloring *if and only if $(\tilde{G}, S)$ is a* yes *instance of* Steiner Rainbow k-Coloring.

Proof. In the forward direction, let G' be a *yes* instance of k-Coloring, and $c : V(G') \to [k]$ be one of its solution. We create a coloring $c_R : E(\tilde{G}) \to [k]$ as follows. For $i \in [t]$ and $v \in C'_i$ we set $c_R(c_i, v) = c(v)$. For $i, j \in [t]$, $i \neq j$ let u, v be the (unique) vertices in C'_i and C'_j such that $(u, v) \in E(G')$. We now describe the value of c_R for edges in the path $P = (u, x_1^{uv}, \cdots, x_{k-3}^{uv}, v)$. Notice that $|E(P)| = k - 2$ therefore, we arbitrarily assign distinct integers in $[k] \setminus \{c_R(c_i, u), c_R(c_j, v)\}$ to $c_R(e)$, where $e \in E(P)$. Since c is a proper coloring of G' therefore, $c_R(c_i, u) = c(u) \neq c(v) = c_R(c_j, v)$. This together with the definition of c_R for edges in P implies that there is a rainbow path, namely $(c_i, u, x_1^{uv}, \cdots, x_{k-3}^{uv}, v, c_j)$ in $\tilde{G}$ between c_i and c_j. This concludes the proof in the forward direction.

In the reverse direction, let $(\tilde{G}, S)$ be a *yes* instance of Steiner Rainbow k-Coloring, and $c_R : E(\tilde{G}) \to [k]$ be one of its solution. We create coloring $c : V(G') \to [k]$ as follows. For $i \in [t]$ and $v \in C'_i$, we let $c(v) = c_R(c_i, v)$. We show that c is a solution to k-Coloring in G'. Consider $(u, v) \in E(G')$, and let $u \in C'_i$ and $v \in C'_j$. Note that we have $i \neq j$. Let P be a rainbow path between c_i and c_j in $\tilde{G}$. By the construction of $\tilde{G}$, we have $N_{\tilde{G}}[c_i] \cap N_{\tilde{G}}[c_j] = \emptyset$. Moreover, since P is a rainbow path therefore, it can contain at most k edges. Since $N_{\tilde{G}}(c_i) = C'_i$ and $N_{\tilde{G}}(c_j) = C'_j$, and there is a exactly one path with at most $k - 2$ edges between a vertex in C'_i and a vertex in C'_j, namely $(c_i, u, x_1^{uv}, \cdots, x_{k-3}^{uv}, v, c_j)$. Therefore, by construction of c together with the fact that P is a rainbow path we have $c(u) \neq c(v)$. This concludes the proof. ◀

▶ **Theorem 2.** Steiner Rainbow k-Coloring *does not admit an algorithm running in time $2^{o(|S|^2)} n^{\mathcal{O}(1)}$, unless* ETH *fails. Here, n is the number of vertices in the input graph.*

Proof. Follows from construction of an instance $(\tilde{G}, S)$ with $|S| \in \mathcal{O}(\sqrt{n})$ of Steiner Rainbow k-Coloring for a given instance G of k-Coloring with maximum degree at most $2(k - 2)$, Lemma 1, and existence of no algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ for k-Coloring on graphs of maximum degree $2(k - 2)$ (assuming ETH). ◀

4 Lower bound for Rainbow k-Coloring

In this section, we show that for every $k \geq 3$, Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, unless ETH fails. We give different reductions for the case when $k = 3$ (Section 4.1), k is an even number greater than 3 (Section 4.2), and k is an odd number greater than 4 (Section 4.3). We note that although the approach used for proving lower bound for Rainbow 3-Coloring is extensible to Rainbow k-Coloring when k is odd, but it unnecessarily adds to complexity of the reduction. Moreover, the approach we follow for showing the lower bound result for $k > 3$, where k is an odd number introduces some technical issues when we try to extend it for $k = 3$.

Towards proving our lower bound result, we give an appropriate reduction from k-Coloring on graphs of maximum degree $2(k-1)$, which does not admit an algorithm running in time $2^{o(n)}n^{\mathcal{O}(1)}$ unless ETH fails. The key idea behind the reduction is same as that presented in Section 3, but for this case it is more involved. Before moving on to the description of the reductions we define a graph that will be useful in our reductions.

A *clique sequence* $\mathbb{Z}_{n,t} = (Z_1, Z_2, \cdots Z_t)$ of order (n,t) is a graph defined as follows. We have $V(\mathbb{Z}_{k,t}) = \uplus_{i\in[t]} Z_i$, where $|Z_i| = n$ for all $i \in [t]$. For each $i \in [t]$, all the edges in $\{(z,z') \mid z, z' \in Z_i\}$ are present in $E(\mathbb{Z}_{n,t})$, *i.e.* Z_i is a clique. Furthermore, for all $i \in [t-1]$ all the edges in $\{(z,z') \mid z, \in Z_i, x' \in Z_{i+1}\}$ are present in $E(\mathbb{Z}_{n,t})$. These are exactly edges in $E(\mathbb{Z}_{n,t})$.

4.1 Lower bound for Rainbow 3-Coloring

In this section, we show that Rainbow 3-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, where n is the number of vertices in the input graph G.

Let G be an instance of 3-Coloring on n vertices with maximum degree bounded by 4. We start by computing (in polynomial time) a harmonious coloring φ of G with $t \in \mathcal{O}(\sqrt{n})$ color classes such that each color class contains at most $\mathcal{O}(\sqrt{n})$ vertices [11, 15, 24, 28]. Let $C_1, C_2, \cdots, C_t$ be the color classes of φ. From the discussion in Section 3, we assume that for $i, j \in [t]$, $i \neq j$ there is exactly one edge between C_i and C_j in G. We construct an instance G' of Rainbow 3-Coloring as follows (see Figure 1).

- *Color class gadget.* Consider $i \in [t]$. The color class gadget $\mathcal{C}_i$ comprises of the set C_i, two vertices c_i, b_i, and a clique U_i on 3 vertices with vertex set as $\{u^i_1, u^i_2, u^i_3\}$. We add all the edges in $\{(v, c_i), (v, b_i), (v, u^i_1), (v, u^i_2), (v, u^i_3) \mid v \in C_i\}$ to $E(\mathcal{C}_i)$. Also, we add the edge (b_i, c_i) to $E(\mathcal{C}_i)$.
- *Connection between color class gadgets.* Consider $i, j \in [t]$, $i \neq j$ we add all the edges in $\{(b_i, u^j_\ell) \mid \ell \in [3]\}$ to $E(G')$. Furthermore, we add all the edges $\{(u^i_\ell, u^j_{\ell'}) \mid \ell, \ell' \in [3]\}$ to $E(G')$. Note that $\{u^{i'}_\ell \mid i' \in [t], \ell \in [3]\}$ induces a clique in G'.
- *Encoding edges.* For this case encoding edges is quite simple. For $i, j \in [t]$, $i \neq j$ we add the unique edge (u, v) between C_i and C_j with $u \in C_i$ and $v \in C_j$ to G'. Note that this is same as adding all the edges in $E(G)$ to $E(G')$.

This finishes the description of the instance G' of Rainbow 3-Coloring. We note that some of the edges in G' are not necessary for the correctness of the reduction. However, they are added to reduce the number of pairs for which we need to argue about existence of a rainbow path. Before moving on to the proof of equivalence between these instances, we create an edge coloring function $c_R : E(G') \to [3]$. Here, we create c_R based on a solution c to 3-Coloring in G, assuming that G is a *yes* instance of 3-Coloring. We will follow computation modulo k, and therefore color 0 is same as color k.

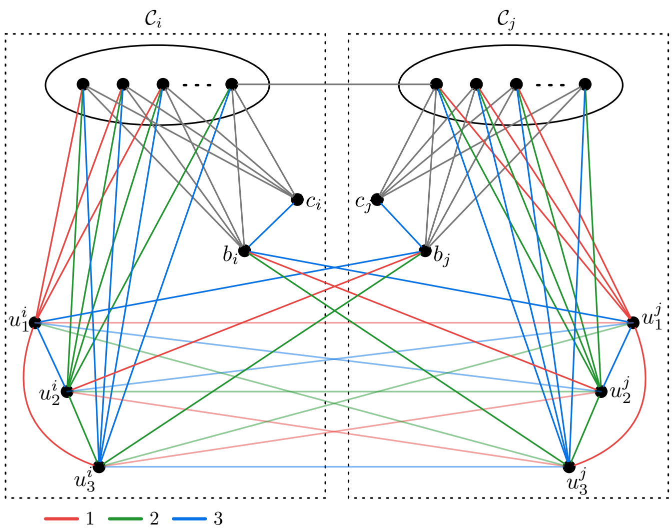

Figure 1 An illustration of (partial) construction of the graph G' and the coloring function c_R.

▶ **Definition 3.** Given a solution c to 3-Coloring in G, we construct $c_R : E(G') \to [3]$ as follows (see Figure 1).

1. For $i \in [t]$, and $v \in C_i$ set $c_R(v, c_i) = c(v)$, $c_R(v, b_i) = c(v)$, and for $\ell \in [3]$, $c_R(v, u^i_\ell) = \ell$.
2. For $i, j \in [t]$, $i \neq j$ let (u, v) be the unique edge between C_i and C_j. We set $c_R(u, v)$ to be the unique integer in $[3] \setminus \{c(u), c(v)\}$. Here, the uniqueness is guaranteed by the fact that c is a 3-Coloring of G, and $(u, v) \in E(G')$ therefore, $c(u) \neq c(v)$.
3. For $i \in [t]$ set $c_R(b_i, c_i) = 3$, $c_R(u^i_1, u^i_2) = 3$, $c_R(u^i_2, u^i_3) = 2$, and $c_R(u^i_3, u^i_1) = 1$.
4. For $i, j \in [t]$, $i \neq j$ and $\ell \in [3]$ set $c_R(b_i, u^j_\ell) = \ell - 1$.
5. For $i, j \in [t]$, $i \neq j$ and $\ell \in [3]$ set $c_R(u^i_\ell, u^j_\ell) = \ell$. Furthermore, for $\ell' \in [3] \setminus \{\ell\}$ we set $c_R(u^i_\ell, u^j_{\ell'}) = \hat{\ell}$, where $\hat{\ell}$ is the unique integer in $[3] \setminus \{\ell, \ell'\}$.

Next, we prove some lemmata that will be useful in establishing the equivalence between the instance G of 3-Coloring and the instance G' of Rainbow 3-Coloring.

▶ **Lemma 4.** *For $i, j \in [t]$, where $i \neq j$ let (u^*, v^*) be the unique edge between C_i and C_j with $u^* \in C_i$ and $v^* \in C_j$. There is exactly one path, namely (c_i, u^*, v^*, c_j) in G' between c_i and c_j that has at most 3 edges.*

Proof. Consider $i, j \in [t]$, where $i \neq j$. Let $u^* \in C_i$, $v^*_j \in C_j$ be the vertices such that $(u^*, v^*) \in E(G')$. Recall that $N(c_i) = \{b_i\} \cup C_i$ and $N(c_j) = \{b_j\} \cup C_j$. Therefore, any path between c_i and c_j with at most 3 edges must contain a vertex $u \in N(c_i) \cup \{b_i\}$ and a vertex $v \in N(c_j) \cup \{b_j\}$ such that $(u, v) \in E(G')$. Observe that $(b_i, b_j) \notin E(G')$, $b_i \notin N(C_j)$, and $b_j \notin N(C_i)$. Therefore, u must belong to C_i and v must belong to C_j. But there is unique edge between C_i and C_j, namely (u^*, v^*). Therefore, $u = u^*$ and $v = v^*$. This concludes the proof. ◀

▶ **Lemma 5.** *Let G be a yes instance of 3-Coloring, and c be one of its solution. Furthermore, let $c_R : E(G') \to [3]$ be the coloring given by Definition 3 for the coloring c of G. Then for all $i \in [t]$, and $u, v \in \mathcal{C}_i$ there is a rainbow path between u and v in G'.*

Proof. Consider $i \in [t]$. Recall that $V(\mathcal{C}_i) = C_i \cup \{c_i, b_i, u^i_1, u^i_2, u^i_3\}$. We will argue for the pairs $(u, v) \in V(\mathcal{C}_i) \times V(\mathcal{C}_i)$ such that $(u, v) \notin E(\mathcal{C}_i)$, since we trivially have a rainbow path between pair of vertices that have an edge between them. Therefore, we argue about pairs in $\{(u,v) \mid u,v \in C_i, u \neq v\} \cup \{(b_i,u^i_\ell),(c_i,u^i_\ell) \mid \ell \in [3]\}$.

- Consider $u,v \in C_i$ where $u \neq v$. The path $(u \overset{1}{—} u^i_1 \overset{3}{—} u^i_2 \overset{2}{—} v)$ is a rainbow path between u and v in G'.
- Consider $v \in C_i$. If $c_R(v,b_i) = 1$ then $(b_i \overset{1}{—} v \overset{2}{—} u^i_2 \overset{3}{—} u^i_1)$ is a rainbow path between b_i and u^i_1, $(b_i \overset{1}{—} v \overset{2}{—} u^i_2)$ is a rainbow path between b_i and u^i_2, and $(b_i \overset{1}{—} v \overset{3}{—} u^i_3)$ is a rainbow path between b_i and u^i_3. All other cases can be argued analogously. Also, similar arguments can be given for rainbow paths between c_i and vertices in $\{u^i_\ell \mid \ell \in [3]\}$.

◀

▶ **Lemma 6.** *Let G be a yes instance of* 3-Coloring*, and c be one of its solution. Furthermore, let $c_R : E(G') \to [3]$ be the coloring given by Definition 3 for the coloring c of G. Then for all $i,j \in [t]$, $i \neq j$ for all $u \in \mathcal{C}_i$ and $v \in \mathcal{C}_j$ there is a rainbow path between u and v in G'.*

Proof. Consider $i,j \in [t]$, where $i \neq j$. Let $(u^*,v^*) \in E(G')$ be the unique edge between C_i and C_j with $u^* \in C_i$ and $v^* \in C_j$. We will argue for the pairs $(u,v) \in V(\mathcal{C}_i) \times V(\mathcal{C}_j)$ such that $(u,v) \notin E(G')$, since we trivially have a rainbow path between pair of vertices that have an edge between them. Therefore, we argue only for pairs in the following sets.

- $A_1 = \{(c_i,u) \mid v \in \{b_j,c_j\} \cup C_j \cup \{u^j_\ell \mid \ell \in [3]\}\}$.
- $A_2 = \{(b_i,u) \mid v \in \{b_j\} \cup C_j\}$.
- $A_3 = \{(u,v) \mid v \in C_i, v \in C_j \cup \{u^j_\ell \mid \ell \in [3]\}\}$.

Although, $A_1 \cup A_2 \cup A_3$ does not include all the pairs in $(V(\mathcal{C}_i) \times V(\mathcal{C}_j)) \setminus E(G')$, but a rainbow path for all such pairs can be obtained by following a symmetric argument (swapping i and j). We now show that for each pair in $A_1 \cup A_2 \cup A_3$ we have a rainbow path between them in G'.

- The path (c_i,u^*,v^*,c_j) is a rainbow path between c_i and c_j in G' (see item 1 and 2 of Definition 3). Similarly, (c_i,u^*,v^*,b_j) is a rainbow path between c_i and b_j in G'. Consider a vertex $v \in C_j$. The path $(c_i \overset{3}{—} b_i \overset{1}{—} u^j_2 \overset{2}{—} v)$ is a rainbow path between c_i and v (see item 1, 3, and 4 of Definition 3). This also gives a rainbow path between c_i and u^j_2. The paths $(c_i \overset{3}{—} b_i \overset{2}{—} u^j_3)$ and $(c_i \overset{3}{—} b_i \overset{2}{—} u^j_3 \overset{1}{—} u^j_1)$ are rainbow paths between c_i and u^j_3, and between c_i and u^j_1, respectively (see item 3 and 4 of Definition 3).
- The path (b_i,u^*,v^*,b_j) is a rainbow path between b_i and b_j in G' (see item 1 and 2 of Definition 3). For $v \in C_j$, $(b_i \overset{1}{—} u^j_2 \overset{2}{—} v)$ is a rainbow path between b_i and v (see item 1 and 4 of Definition 3).
- Consider a vertex $u \in C_i$. For $v \in C_j$, $(u \overset{1}{—} u^i_1 \overset{3}{—} u^j_2 \overset{2}{—} v)$ is a rainbow path between u and v in G' (see item 1 and 5 of Definition 3). Note that this also gives a rainbow path between u and u^j_2. The path $(u \overset{1}{—} u^i_1 \overset{2}{—} u^j_3)$ is a rainbow path between u and u^j_3. Finally, the path $u \overset{3}{—} u^i_3 \overset{2}{—} u^j_1$ is a rainbow path between u and u^j_1.

◀

We now establish equivalence between the instance G of 3-Coloring and the instance G' of Rainbow 3-Coloring.

▶ **Lemma 7.** *G is a* yes *instance of* 3-Coloring *if and only if G' is a* yes *instance of* Rainbow 3-Coloring*.*

Proof. In the forward direction, let G be a *yes* instance of 3-Coloring, and $c : V(G) \to [3]$ be one of its solution. Let $c_R : E(G') \to [3]$ be the coloring given by Definition 3 for the given coloring c of G. From Lemma 5 and 6 it follows that c_R is a solution to Rainbow 3-Coloring in G'.

In the reverse direction, let G' be a *yes* instance of Rainbow 3-Coloring, and $c_R : E(G') \to [3]$ be one of its solution. We create coloring $c : V(G) \to [3]$ as follows. For $i \in [t]$ and $v \in C_i$, we let $c(v) = c_R(c_i, v)$. We show that c is a valid solution to 3-Coloring in G. Consider $(u, v) \in E(G)$, and let $u \in C_i$ and $v \in C_j$. Note that we have $i \neq j$. Let P be a rainbow path between c_i and c_j in G'. Note that P can have at most 3 edges. By Lemma 4 we know that $P = (c_i, u, v, c_j)$, therefore by construction of c, we have $c_R(c_i, u) = c(u) \neq c(v) = c_R(c_i, v)$. This concludes the proof. ◀

▶ **Theorem 8.** Rainbow 3-Coloring *does not admit an algorithm running in time* $2^{o(|E(G)|)} n^{\mathcal{O}(1)}$*, unless* ETH *fails. Here, n is the number of vertices in the input graph.*

Proof. Follows from construction of an instance G' of Rainbow 3-Coloring with $|E(G')| \in \mathcal{O}(|V(G)|)$ for a given instance G of 3-Coloring with maximum degree bounded by 4, Lemma 7, and existence of no algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ for 3-Coloring on graphs of maximum degree 4 (assuming ETH). ◀

4.2 Lower Bound for Rainbow k-Coloring, $k > 3$ and even

In this section, we show that Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)} n^{\mathcal{O}(1)}$, for every even k where $k > 3$. Here, n is the number of vertices in the input graph.

Let G be an instance of k-Coloring on n vertices with maximum degree bounded by $2(k-1)$. Here, $k > 3$ and k is an even number. We start by computing (in polynomial time) a harmonious coloring φ of G with $t \in \mathcal{O}(\sqrt{n})$ color classes such that each color class contains at most $\mathcal{O}(\sqrt{n})$ vertices [11, 15, 24, 28]. Let $C_1, C_2, \cdots, C_t$ be the color classes of φ with exactly one edge between C_i and C_j in G, where $i, j \in [t]$. We modify the graph G and its harmonious coloring φ, to obtain a more structured instance, which will be useful later. For each $i \in [t]$, we add k new vertices $v^*_{i1}, v^*_{i2}, \cdots, v^*_{ik}$ to $V(G)$, and add them to C_i. We continue to call the modified graph as G and its harmonious coloring as φ with color classes $C_1, C_2, \cdots, C_t$. We note that $\{v^*_{ij} \mid i \in [t], j \in [k]\}$ induce an independent set in G. The purpose of adding these k new vertices is to ensure that if G is a *yes* instance of k-Coloring then there is a k-coloring c of G, such that for each $i \in [t]$ and $j \in [k]$, we have $c^{-1}(j) \cap C_i \neq \emptyset$. This will helpful in simplifying some of the arguments later. Observe that original instance is a *yes* instance of a k-Coloring is and only if the modified instance is a *yes* instance of k-Coloring. Moreover, given a k-coloring of G (modified graph), in polynomial time we can obtain another k-coloring c' of G such that for all $i \in [t], j \in [k]$ we have $c(v^*_{ij}) = j$. Also, we have $|V(G)| \in \mathcal{O}(n)$, and $|E(G)| \in \mathcal{O}(n)$, where n is the number of vertices in the original instance. Hereafter, whenever we talk about a solution c to k-Coloring in G (if it exists) we will assume (without explicitly mentioning) that for all $i \in [t]$ and $p \in [k]$ we have $C_i \cap c^{-1}(p) \neq \emptyset$. We now move to description of the reduction.

We proceed by describing color class gadget $\mathcal{C}_i$, corresponding to the color class C_i, where $i \in [t]$, and gadgets to encode edges in G. Then we state the connection between various color class gadgets and edge gadgets. We let $k = 2\ell$, where $\ell \in \mathbb{N}$ and $\ell > 1$. We create an instance G' of Rainbow k-Coloring as described below (see Figure 2).

- *Color class gadget.* Consider $i \in [t]$. The color class gadget $\mathcal{C}_i$ comprises of the set C_i, a vertex c_i, and a clique sequence $\mathbb{Z}_i = (U^i_1 \cup D^i_1, \cdots, U^i_{\ell-1} \cup D^i_{\ell-1})$ of order $(2k, \ell-1)$. Here,

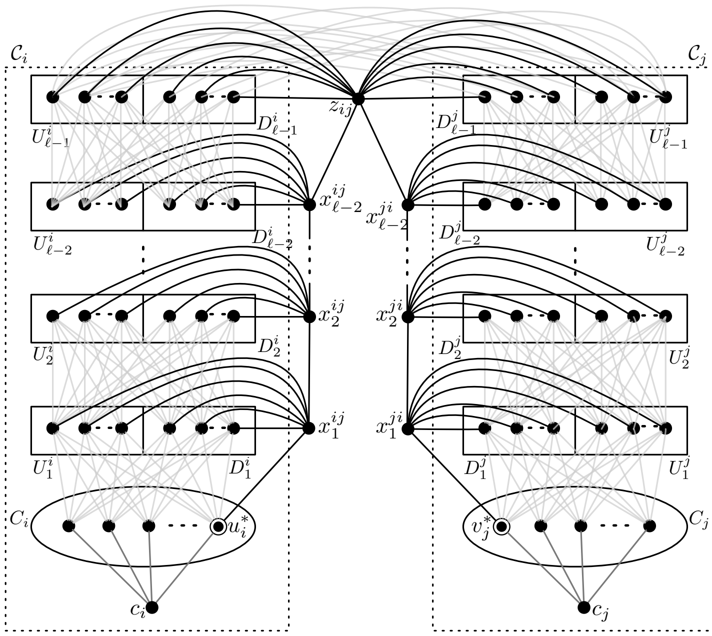

Figure 2 An illustration of (partial) construction the instance G' of k-Coloring, where $k > 3$ and k is even.

for each $i \in [\ell-1]$ we have $|U_i| = |D_i| = k$. For $r \in [\ell-1]$ we let $U^i_r = \{u^i_{rp} \mid p \in [k]\}$, and $D^i_r = \{d^i_{rp} \mid p \in [k]\}$. We add all the edges in $\{(c_i, v) \mid v \in C_i\}$ to $E(\mathcal{C}_i)$. Also, we add all the edges in $\{(v, w) \mid v \in C_i, w \in U^i_1 \cup D^i_1\}$ to $E(\mathcal{C}_i)$.

- *Connection between color class gadgets.* For each $i, j \in [t]$ where $i \neq j$, we add all the edges in $\{(w, w') \mid w \in U^i_{\ell-1} \cup D^i_{\ell-1}, w' \in U^j_{\ell-1} \cup D^j_{\ell-1}\}$ to $E(G')$.
- *Edge gadget.* Consider $i, j \in [t]$ with $i < j$. Recall that there is exactly one edge between C_i and C_j. Corresponding to this edge we create a path $P = (x^{ij}_1, \cdots, x^{ij}_{\ell-2}, z_{ij}, x^{ji}_{\ell-2}, \cdots, x^{ji}_1)$ on $k-3$ vertices, and add it to G'. We note that whenever we say vertex z_{ji} it refers to the vertex z_{ij} *i.e.* z_{ij} and z_{ji} denotes the same vertex.
- *Connection between color class gadgets and edge gadgets.* Consider $i, j \in [t]$, where $i < j$. Let (u^*_i, v^*_j) be the unique edge between C_i and C_j with $u^*_i \in C_i$ and $v^*_j \in C_i$. We add the edges $(u^*_i, x^{ij}_1), (x^{ji}_1, v^*_j)$ to $E(G')$. Notice that when $\ell = 2$ x^{ij}_1 does not exists. In this case, we add the edges $(u^*_i, z), (z, v^*_j)$ to $E(G')$. For each $r \in [\ell-2]$ we add all the edges in $\{(x^{ij}_r, w) \mid w \in U^i_r \cup D^i_r\}$ to $E(G')$. Similarly, we add all the edges in $\{(x^{ji}_r, w) \mid w \in U^j_r \cup D^j_r\}$ to $E(G')$. Also, we add all the edges in $\{(z_{ij}, u) \mid u \in U^i_{\ell-1} \cup D^i_{\ell-1} \cup U^j_{\ell-1} \cup D^j_{\ell-1}\}$ to $E(G')$.

This finishes the construction of instance G' of Rainbow k-Coloring for the given instance G of k-Coloring. Before moving on to proving the equivalence between these instances, we create an edge coloring function $c_R : E(G') \to [k]$. Here, we create c_R based on a solution c to k-Coloring in G, assuming that is G a *yes* instance of k-Coloring. We will follow computation modulo k (color 0 is same as color k).

▶ **Definition 9.** Given a solution c to k-Coloring in G, we construct $c_R : E(G') \to [k]$ as follows.

1. For $i \in [t]$, and $v \in C_i$ we set $c_R(v, c_i) = c(v)$.

2. For $i, j \in [t]$, $i < j$ let (u_i^*, v_j^*) be the unique edge between C_i and C_j. Consider the path $P = (u_i^*, x_1^{ij}, \cdots x_{\ell-2}^{ij}, z_{ij}, x_{\ell-2}^{ji}, \cdots x_1^{ji}, v_j^*)$. We arbitrarily assign unique integers in $[k] \setminus \{c(u_i^*), c(v_j^*)\}$ to $c_R(e)$, for each $e \in E(P)$.
3. For $i \in [t]$, a vertex $v \in C_i$, and $p \in [k]$ we set $c_R(v, u_{1p}^i) = p - 1$, and $c_R(v, d_{1p}^i) = p$.
4. For $i \in [t]$, $r \in [\ell - 1]$, and $p, q \in [k]$ we set $c_R(d_{rp}^i, u_{rq}^i) = p$.
5. For $i, j \in [t]$, where $i \neq j$, $r \in [\ell - 1]$, and $p \in [k]$ we set $c_R(x_r^{ij}, u_{rp}^i) = p$, and $c_R(x_r^{ij}, d_{rp}^i) = p + 1$.
6. For $i \in [t]$, $r \in [\ell - 2]$, $p, q \in [k]$ we set $c_R(d_{(r+1)p}^i, d_{rq}^i) = p$, and $c_R(u_{rp}^i, u_{(r+1)q}^i) = p$.
7. For $i, j \in [t]$ where $i \neq j$, $p, q \in [k]$ we set $c_R(u_{(\ell-1)p}^i, d_{(\ell-1)q}^j) = p$, $c_R(u_{(\ell-1)p}^i, z_{ij}) = p$, and $c_R(d_{(\ell-1)p}^i, z_{ij}) = p + 1$.
8. For $i \in [t]$, $r \in [\ell - 2]$, $p, q \in [k]$ we set $c_R(u_{rp}^i, d_{(r+1)q}^i) = q$ and $c_R(u_{(r+1)p}^i, d_{rq}^i) = p$.
9. For all $i \in [t]$, $r \in [\ell - 1]$, $p, q \in [k]$, where $p \neq q$ we set $c_R(u_{rp}^i, u_{rq}^i) = k$.
10. For all the remaining edges in $E(G')$, c_R assigns it an integer in $[k]$ arbitrarily.

For a vertex $v \in V(G')$, by T_v we denote the breadth first search tree in G' with v as the root vertex. We let $L_0^v = \{v\}$. For $i \in [n']$, by L_i^v we denote the set of vertices which are at a distance i from v in T_v. Here, the distance between $u \in V(G')$ and v denotes the number of edges in the unique path between v and u in T_v and $n' = |V(G')|$.

Next, we prove some lemmata that will be useful in establishing equivalence between the instance G of k-Coloring and the instance G' of Rainbow k-Coloring.

▶ **Lemma 10.** *For $i, j \in [t]$, where $i \neq j$, let P be a path between c_i and c_j with at most k edges in G'. If $\ell > 2$ then P contain the edge $(x_{\ell-2}^{ij}, z_{ij})$. Otherwise, P contains the edge (u, z_{ij}), where u is the unique vertex in C_i that in adjacent to a vertex in C_j.*

Proof. Consider $i, j \in [t]$, where $i \neq j$. Let P be a path between c_i and c_j with at most k edges in G'. Recall that $N(c_i) = C_i$ and $N(c_j) = C_j$, where $C_i \cap C_j = \emptyset$. Therefore, P must contain an edge (c_i, u) and (v, c_j), where $u \in C_i$ and $v \in C_j$ $(u \neq v)$. Consider the breadth first search tree T_{c_i}. We start by looking at first $\ell - 1$ levels of trees T_{c_i} and T_{c_j} (starting from 0). Notice that for $r \in [\ell - 2]$ we have $L_{r+1}^{c_i} = U_r^i \cup D_r^i \cup \{x_r^{ij'} \mid j' \in [t] \setminus \{i\}\}$, and $L_1^{c_i} = C_i$. Similarly, for $r \in [\ell - 2]$ we have $L_{r+1}^{c_j} = U_r^j \cup D_r^j \cup \{x_r^{ji'} \mid i' \in [t] \setminus \{j\}\}$, and $L_1^{c_j} = C_j$. For all $r, r' \in [\ell - 1]$ we have $L_r^{c_i} \cap L_{r'}^{c_j} = \emptyset$. For each $w \in \{c_i, c_j\}$ and $r \in [\ell]$, P must contain a vertex from L_r^w. But then P can contain at most one other vertex. Recall that for all $r, r' \in [\ell - 1]$, there is no edge between a vertex in $L_r^{c_i}$ and a vertex in $L_{r'}^{c_j}$. Therefore, P must contain exactly one other vertex, which belongs to $L_\ell^{c_i} \cap L_\ell^{c_j}$. But $L_\ell^{c_i} \cap L_\ell^{c_j} = \{z_{ij}\}$. Therefore, P must contain the vertex z_{ij}. Notice that P must contain exactly one vertex from each L_r^w, where $w \in \{c_i, c_j\}$ and $r \in [\ell - 1]$. Moreover, the only vertex adjacent to z_{ij} in $L_0^{c_i} \cup (\cup_{r \in [\ell-1]} L_r^{c_i})$ is $x_{\ell-2}^{ij}$. Therefore, P must contain the edge $(x_{\ell-2}^{ij}, z_{ij})$. We note here that when $\ell = 2$, then $x_{\ell-2}^{ij}$ is same as the vertex unique vertex $u \in C_i$, which is adjacent to a vertex in C_j. ◀

▶ **Lemma 11.** *For $i, j \in [t]$, where $i \neq j$ let (u^*, v^*) be the unique edge between C_i and C_j with $u^* \in C_i$ and $v^* \in C_j$. There is exactly one path, namely $(c_i, u^*, x_1^{ij}, \cdots, x_{\ell-2}^{ij}, z_{ij}, x_{\ell-2}^{ji}, \cdots, x_1^{ji}, v^*, c_j)$ in G' between c_i and c_j that has at most k edges.*

Proof. Consider $i, j \in [t]$, where $i \neq j$. Let $u^* \in C_i$, $v_j^* \in C_j$ be the vertices such that $(u^*, v^*) \in E(G')$. Also, let P be a (simple) path between c_i and c_j with at most k edges in G'. By construction of G', P contains an edge (c_i, u) and an edge (v, c_j), where $u \in C_i$ and $v \in C_j$, respectively. Recall that for $r \in [\ell - 2]$ we have $L_{r+1}^{c_i} = U_r^i \cup D_r^i \cup \{x_r^{ij'} \mid j' \in [t] \setminus \{i\}\}$, $L_1^{c_i} = C_i$, $L_{r+1}^{c_j} = U_r^j \cup D_r^j \cup \{x_r^{ji'} \mid i' \in [t] \setminus \{i\}\}$, and $L_1^{c_j} = C_j$. Moreover, for all $r, r' \in [\ell - 1]$

we have $L_r^{c_i} \cap L_{r'}^{c_j} = \emptyset$, and there is no edge between a vertex in $L_r^{c_i}$ and a vertex in $L_{r'}^{c_j}$. From Lemma 10 we know that P contains the vertex z_{ij}. If $\ell = 2$, then the claim trivially follows. Otherwise, for each $w \in \{c_i, c_j\}$ and $r \in [\ell-1]$, P contains exactly one vertex from L_r^w. Also, from Lemma 10 we know that $(x_{\ell-2}^{ij}, z_{ij}), (z_{ij}, x_{\ell-2}^{ji}) \in E(P)$. Therefore, P either contains a sub-path P_1 from c_i to $x_{\ell-2}^{ji}$ and a sub-path P_2 from $x_{\ell-2}^{ij}$ to c_j or it contains a sub-path P_1' from c_i to $x_{\ell-2}^{ij}$ and a sub-path P_2' from $x_{\ell-2}^{ji}$ to c_j. Consider the case when P contains a sub-path P_1 from c_i to $x_{\ell-2}^{ji}$ and a sub-path P_2 from $x_{\ell-2}^{ij}$ to c_j. Since P is simple path therefore, $E(P_1) \cap E(P_2) = \emptyset$, and $V(P_1) \cap V(P_2) = \emptyset$. Moreover, any path from c_i to $x_{\ell-2}^{ji}$ contains at least ℓ edges. This is implied from the fact that $x_{\ell-2}^{ji} \in L_{\ell+1}^{c_i}$. Similarly, any path from c_j to $x_{\ell-1}^{ij}$ contains at least ℓ edges. But then P contains at least $2\ell+1 > k$ edges.

Next, consider the case when P contains a sub-path P_1' from c_i to $x_{\ell-2}^{ij}$ and a sub-path P_2' from $x_{\ell-2}^{ji}$ to c_j. Notice that the shortest path from c_i to $x_{\ell-1}^{ij}$ has at least $\ell-1$ edges. This follows from the fact that $x_{\ell-2}^{ij} \in L_{\ell-1}^{c_i}$. Similarly, the shortest path from $x_{\ell-2}^{ji}$ to c_j has at least $\ell-1$ edges. This implies that P_1' and P_2' both have exactly $\ell-1$ edges. We now show that $P_1' = (c_i, u^*, x_1^{ij}, \cdots, x_{\ell-2}^{ij})$. Consider the smallest number $r \in [\ell-2]$ such that $x_{r-1}^{ij} \notin V(P_1')$ and $x_r^{ij} \in V(P_1')$. Here, for $r = 1$ we assume that $x_{r-1}^{ij} = u^*$. If such an r does not exists then we have $P_1' = (c_i, u^*, x_1^{ij}, \cdots, x_{\ell-2}^{ij})$. This follows from the fact that $x_{\ell-2}^{ij} \in V(P_1')$, and the unique vertex in C_i that is adjacent to x_1^{ij} is u^*. We now consider the case when such an r exists. Recall that for each $r' \in [\ell-1]$ we have $|V(P_1') \cap L_{r'}^{c_i}| = 1$. Therefore, there exists $x \in L_{r-1}^{c_i} \cap V(P_1')$. By construction of G' (and r), we have $(x, x_r^{ij}) \notin E(G')$. This together with the fact that for each $r' \in [\ell-1]$ we have $|V(P_1') \cap L_{r'}^{c_i}| = 1$ implies that such an r does not exist. This concludes the proof. ◀

▶ **Lemma 12.** *Let G be a yes instance of k-*Coloring*, and c be one of its solution. Furthermore, let $c_R : E(G') \to [k]$ be the coloring given by Definition 9 for the coloring c of G. For all $i \in [t]$, and $u, v \in V(\mathcal{C}_i) \cup \{z_{ij} \mid j \in [k] \setminus \{i\}\} \cup \{x_r^{ij} \mid j \in [t] \setminus \{i\}, r \in [\ell-2]\}$ there is a rainbow path between u and v in G'.*

Proof. Consider $i \in [t]$. Recall that $V(\mathcal{C}_i) = \{c_i\} \cup C_i \cup \{u_{rp}^i, d_{rp}^i \mid r \in [\ell-1], p \in [k]\}$. Let $U_i = \cup_{r\in[\ell-1]} U_r^i$, $D_i = \cup_{r\in[\ell-1]} D_r^i$, $X_i = \{x_r^{ij} \mid j \in [t] \setminus \{i\}, r \in [\ell-2]\}$, and $Z = \{z_{ij} \mid j \in [k] \setminus \{i\}\}$. We consider pairs of vertices in the following sets.

- $A_1 = \{(c_i, v) \mid v \in U_i \cup D_i \cup X_i \cup Z\}$.
- $A_2 = \{(u, v) \mid u \in C_i, v \in (U_i \setminus U_1^i) \cup (D_i \setminus D_1^i) \cup X_i \cup Z\}$.
- $A_3 = \{(u, v) \mid u \neq v, u \in U_i, v \in U_i \cup D_i \cup X_i \cup Z\}$.
- $A_4 = \{(u, v) \mid u \neq v, u \in D_i, v \in D_i \cup X_i \cup Z\}$.
- $A_5 = \{(u, v) \mid u \neq v, u \in X_i, v \in X_i \cup Z\}$

We now show that each pair in $\cup_{r\in[5]} A_r$ has a rainbow path between them. We will argue only about non-adjacent pairs of vertices.

- For each $x \in X_i \cup Z$, by construction of c_R (and G') it follows that there is a rainbow path between c_i and x (see item 1 and 2 of Definition 9). For $p \in [k]$, let $v_p^* \in C_i$ be a vertex such that $c_R(c_i, v_p^*) = p$. For $p \in [k]$ the path $(c_i \overset{p}{—} v_p^* \overset{p-1}{—} u_{1p}^i)$ is a rainbow path between c_i and u_{1p}^i (see item 1 and 3 of Definition 9). Similarly, $(c_i \overset{p+1}{—} v_{p+1}^* \overset{p}{—} d_{1p}^i)$ is a rainbow path between c_i and d_{1p}^i. For $r \in [\ell-1] \setminus \{1\}$ and $p \in [k]$ the path $(c_i \overset{k-1}{—} v_{k-1}^* \overset{k}{—} u_{11}^i \overset{1}{—} u_{22}^i \cdots\ u_{(r-1)(r-1)}^i \overset{r-1}{—} u_{rp}^i)$ is a rainbow path between c_i and u_{rp}^i in G' (see item 1, 3, and 6 of Definition 9). Similarly, for $r \in [\ell-1] \setminus \{1\}$, $p \in [k]$ the path

$(c_i \overset{p+r}{—} v^*_{p+r} \overset{p+r-1}{—} d^i_{1(p+r-1)} \overset{p+r-2}{—} d^i_{2(p+r-2)} \cdots\ d^i_{(r-1)(p+1)} \overset{p}{—} d^i_{rp})$ is a rainbow path between c_i and d^i_{rp} in G'.

- Consider $v \in C_i$. For a vertex $z \in Z$, the path $(v \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{\ell-2(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} z)$ is a rainbow path between v and z in G' (see item 3, 6, and 7 of Definition 9). For $r \in [\ell-1]\setminus\{1\}$ and $p \in [k]$ the path $(v \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots\ u^i_{(r-1)(r-1)} \overset{r-1}{—} u^i_{rp})$ is a rainbow path between v and u^i_{rp} in G' (see item 3 and 6 of Definition 9). Similarly, for $r \in [\ell-1] \setminus \{1\}$, $p \in [k]$ the path $(v \overset{p+r-1}{—} d^i_{1(p+r-1)} \overset{p+r-2}{—} d^i_{2(p+r-2)} \cdots\ d^i_{(r-2)(p+2)} \overset{p+1}{—} d^i_{(r-1)(p+1)} \overset{p}{—} d^i_{rp})$ is a rainbow path between v and d^i_{rp} in G'. For x^{ij}_r, where $j \in [t] \setminus \{i\}$ and $r \in [\ell-2]$ the path $(v \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots\ u^i_{(r-1)(r-1)} \overset{r-1}{—} u^i_{rr} \overset{r}{—} x^{ij}_r)$ is a rainbow path between v and x^{ij}_r in G' (see item 3, 5, and 6 of Definition 9).
- Consider a vertex u^i_{rp}, where $r \in [\ell-1]$ and $p \in [k]$. Also, consider a vertex u^i_{sq}, where $s \in [\ell-1] \setminus \{r\}$ and $q \in [k]$. Without loss of generality we assume that $r < s$. The path $(u^i_{rp} \overset{p}{—} u^i_{(r+1)(p+1)} \overset{p+1}{—} \cdots u^i_{(s-1)(p+s-r-1)} \overset{p+s-r-1}{—} u^i_{sq})$ is a rainbow path between u^i_{rp} and u^i_{sq} in G' (see item 6 of Definition 9). Consider a vertex d^i_{sq}, where $s \in [\ell-1]\setminus\{r\}$ and $q \in [k]$. If $r < s$ then the path $(d^i_{sq} \overset{q}{—} d^i_{(s-1)(q+1)} \overset{q+1}{—} d^i_{(s-2)(q+2)} \cdots d^i_{(r+1)(q+s-r-1)} \overset{q+s-r-1}{—} u^i_{rp})$ is a rainbow path between u^i_{rp} and d^i_{sq} in G' (see item 6 and 8 of Definition 9). Otherwise, $s < r$ and the path $(d^i_{sq} \overset{q}{—} u^i_{s(q+1)} \overset{q+1}{—} u^i_{(s+1)(q+2)} \overset{q+2}{—} \cdots u^i_{(r-1)(q+s-r)} \overset{q+s-r}{—} u^i_{rp})$ is a rainbow path between d^i_{sq} and u^i_{rp}. Consider a vertex $x^{ij}_s \in X_i$, where $s \in [\ell-1] \setminus \{r\}$. If $r < s$ then the path $(u^i_{rp} \overset{p}{—} u^i_{(r+1)(p+1)} \overset{p+1}{—} \cdots u^i_{s(p+s-r)} \overset{p+s-r}{—} x^{ij}_s)$ is a rainbow path between u^i_{rp} and x^{ij}_s in G' (see item 5 and 6 of Definition 9). Otherwise, $r > s$, and the path $(u^i_{rp} \overset{p}{—} d^i_{(r-1)(p+1)} \overset{p+1}{—} d^i_{(r-2)(p+2)} \cdots d^i_{(s+1)(p+r-s-1)} \overset{p+r-s-1}{—} d^i_{s(p+r-s-1)} \overset{p+r-s}{—} x^{ij}_s)$ is a rainbow path between u^i_{rp} and x^{ij}_s in G' (see item 5, 6, and 8 of Definition 9). Roughly speaking here, using $d^i_{s(p+r-s-1)}$ as a neighbor of x^{ij}_s is just a choice so as to make all the edges in the path to have colors in ascending order. For a vertex $z \in Z$ the path $(u^i_{rp} \overset{p}{—} u^i_{(r+1)(p+1)} \overset{p+1}{—} \cdots u^i_{(\ell-1)(p+\ell-r-1)} \overset{p+\ell-r-1}{—} z)$ is a rainbow path between u^i_{rp} and z in G' (see item 6 and 7 of Definition 9).
- Consider a vertex d^i_{rp}, where $r \in [\ell-1]$ and $p \in [k]$. Next, consider a vertex d^i_{sq}, where $s \in [\ell-1] \setminus \{r\}$ and $q \in [k]$. Without loss of generality we assume $r < s$. The path $(d^i_{rp} \overset{q+s-r-1}{—} d^i_{(r+1)(q+s-r-1)} \cdots d^i_{(s-1)(q+1)} \overset{q}{—} d^i_{sq})$ is a rainbow path between d^i_{rp} and d^i_{sq} in G' (see item 6 of Definition 9). For $z \in Z$ the path $(d^i_{rp} \overset{r+1}{—} d^i_{(r+1)(r+1)} \cdots d^i_{(\ell-2)(\ell-2)} \overset{\ell-1}{—} d^i_{(\ell-1)(\ell-1)} \overset{\ell}{—} z)$ is a rainbow path between d^i_{rp} and z in G' (see item 6 and 7 of Definition 9). For x^{ij}_s, where $s \in [\ell-1] \setminus \{r\}$ consider the following. If $r < s$ then the path $(d^i_{rp} \overset{r+1}{—} d^i_{(r+1)(r+1)} \cdots d^i_{(s-1)(s-1)} \overset{s}{—} d^i_{ss} \overset{s+1}{—} x^{ij}_s)$ is a rainbow path between d^i_{pr} and x^{ij}_s in G' (see item 5 and 6 of Definition 9). Otherwise, $s < r$, and the path $(d^i_{rp} \overset{p}{—} d^i_{(r-1)(p+1)} \overset{p+1}{—} d^i_{(r-2)(p+2)} \cdots d^i_{(s+1)(p+r-s-1)} \overset{p+r-s-1}{—} d^i_{s(p+r-s-1)} \overset{p+r-s}{—} x^{ij}_s)$ is a rainbow path between d^i_{pr} and x^{ij}_s in G' (see item 5 and 6 of Definition 9). Here, we have selected $d^i_{s(p+r-s-1)}$ as the neighbor of x^{ij}_s in the rainbow path we construct instead of $d^i_{s(p+r-s)}$ is just for ensuring that all edges have coloring in ascending order, but we can choose other vertices as well.
- Consider a vertex x^{ij}_r, where $j \in [t] \setminus \{i\}$ and $r \in [\ell-1]$. For all $x \in \{x^{ij}_s \mid s \in [\ell-2]\}$, by the construction of G' and c_R we have a rainbow path between x^{ij}_r and x^{ij}_s (see item 2 in the Definition 9). For a vertex $x^{ij'}_r$, where $j' \in [t] \setminus \{i, j\}$ by construction of c_R the path $(x^{ij}_r, u^i_{r1}, u^i_{r2}, x^{ij'}_r)$ is a rainbow path between x^{ij}_r and $x^{ij'}_r$ (see item 5 and 9 of Definition 9). Next, consider a vertex $x^{ij'}_s$ where $j' \in [t] \setminus \{j\}$ and $s \in [\ell-2] \setminus \{r\}$. Without loss of generality we assume that $r < s$. The path $(x^{ij}_r \overset{r}{—} u^i_{r(r+1)} \overset{r+1}{—} u^i_{(r+1)(r+2)} \cdots u^i_{s(s+1)} \overset{s+1}{—} x^{ij'}_s)$

is a rainbow path between x_r^{ij} and x_s^{ij} (see item 5 and 6 of Definition 9). For $z \in Z$, the path $(x_r^{ij} \overset{r}{—} u^i_{r(r+1)} \overset{r+1}{—} u^i_{(r+1)(r+2)} \cdots u^i_{(\ell-1)\ell} \overset{\ell}{—} z)$ is a rainbow path between x_r^{ij} and z (see item 5, 6, and 7 of Definition 9).

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▶ **Lemma 13.** *Let G be a yes instance of k-*Coloring*, and c be one of its solution. Furthermore, let $c_R : E(G') \to [k]$ be the coloring given by Definition 9 for the coloring c of G. For all $i, j \in [t]$ where $i \neq j$, $u \in V(\mathcal{C}_i) \cup \{z_{ij'} \mid j' \in [k] \setminus \{i\}\} \cup \{x_r^{ij'} \mid j' \in [t] \setminus \{i\}, r \in [\ell-2]\}$ and $v \in V(\mathcal{C}_j) \cup \{z_{ji'} \mid i' \in [k] \setminus \{j\}\} \cup \{x_r^{ji'} \mid i' \in [t] \setminus \{j\}, r \in [\ell-2]\}$ there is a rainbow path between u and v in G'.*

Proof. For $i \in [t]$, let $U_i = \cup_{r \in [\ell-1]} U^i_r$, $D_i = \cup_{r \in [\ell-1]} D^i_r$, $X_i = \{x_r^{ij} \mid j \in [t] \setminus \{i\}, r \in [\ell-1]\}$, and $Z_i = \{z_{ij} \mid j \in [k] \setminus \{i\}\}$. For $i, j \in [t]$, where $i \neq j$ we consider the pairs in the following sets.

- $A_1 = \{(c_i, v) \mid v \in \{c_j\} \cup C_j \cup U_j \cup D_j \cup X_j \cup Z_j\}$.
- $A_2 = \{(u, v) \mid u \in C_i, v \in C_j \cup U_j \cup D_j \cup X_j \cup Z_j\}$.
- $A_3 = \{(u, v) \mid u \in U_i, v \in U_j \cup D_j \cup X_j \cup Z_j\}$.
- $A_3 = \{(u, v) \mid u \in D_i, v \in D_j \cup X_j \cup Z_j\}$.
- $A_3 = \{(u, v) \mid u \in X_i \cup Z_i, v \in X_j \cup Z_j\}$.

Notice that to prove the claim it is enough to show that for every pair in $\cup_{r \in [5]} A_r$ there is a rainbow path between them in G'. For $p \in [k]$, let $v_p^* \in C_i$ be a vertex such that $c_R(c_i, v_p^*) = p$. Next, we show that there is a rainbow path between every pair of vertices in $\cup_{r \in [5]} A_r$.

- Recall that by construction of c_R (and G') we have a rainbow path between c_i and c_j (see item 1 and 2 of Definition 9). For $v \in C_j$ the path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)\ell} \overset{\ell}{—} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{2(2\ell-3)} \overset{2\ell-3}{—} d^j_{1(2\ell-2)} \overset{2\ell-2}{—} v)$ is a rainbow path between c_i and v in G' (see item 1, 3, 6, and 7 of Definition 9). For u^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)\ell} \overset{\ell}{—} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{(r+1)(2\ell-r-2)} \overset{2\ell-r-2}{—} d^j_{r(2\ell-r-1)} \overset{2\ell-r-1}{—} u^j_{rp})$ is a rainbow path between c_i and u^j_{rp} in G' (see item 1, 3, 4, 6, and 7 of Definition 9). For d^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)\ell} \overset{\ell}{—} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{(r+1)(2\ell-r-2)} \overset{2\ell-r-2}{—} d^j_{rp})$ is a rainbow path between c_i and d^j_{rp} in G' (see item 1, 3, 6, and 7 of Definition 9). For x_r^{ji}, where $r \in [\ell-2]$ by construction of c_R and G' we have rainbow path between c_i and x^{ji} (see item 1 and 2 of Definition 9). For $x_r^{ji'}$, where $i' \in [t] \setminus \{i, j\}$ and $r \in [\ell-2]$ consider the path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)\ell} \overset{\ell}{—} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{(r+1)(2\ell-r-2)} \overset{2\ell-r-2}{—} d^j_{r(2\ell-r-1)} \overset{2\ell-r}{—} x_r^{ji'})$ is a rainbow path between c_i and $x_r^{ji'}$ in G' (see item 1, 3, and 5-7 of Definiton 9). For z_{ji} by construction of c_R and G' we have rainbow path between c_i and z_{ji} (see item 1 and 2 of Definition 9). For $z_{ji'}$, where $i' \in [t] \setminus \{i, j\}$ the path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)(\ell-1)} \overset{\ell}{—} z_{ji'})$ is a rainbow path between c_i and $z_{ji'}$ in G' (see item 1, 3, and 5-7 of Definiton 9).
- Consider $u \in C_i$. For $v \in C_j$ the path $(u \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)\ell} \overset{\ell}{—} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{2(2\ell-3)} \overset{2\ell-3}{—} d^j_{1(2\ell-2)} \overset{2\ell-2}{—} v)$ is a rainbow path between u and v in G' (see item 3, 6, and 7 of Definition 9). For u^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(u \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} d^j_{(\ell-1)\ell} \overset{\ell}{—} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{(r+1)(2\ell-r-2)} \overset{2\ell-r-2}{—}$

$d^j_{r(2\ell-r-1)} \overset{2\ell-r-1}{-} u^j_{rp})$ is a rainbow path between u and u^j_{rp} in G' (see item 3, 4, 6, and 7 of Definition 9). For d^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} d^j_{(\ell-1)\ell} \overset{\ell}{-} d^j_{(\ell-2)(\ell+1)} \cdots\ d^j_{(r+1)(2\ell-r-2)} \overset{2\ell-r-2}{-}\ d^j_{rp})$ is a rainbow path between u and d^j_{rp} in G' (see item 3, 6, and 7 of Definition 9). For $x^{ji'}_r$, where $i' \in [t] \setminus \{j\}$ and $r \in [\ell-2]$ the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-}\ d^j_{(\ell-1)\ell} \overset{\ell}{-} d^j_{(\ell-2)(\ell+1)} \cdots\ d^j_{(r+1)(2\ell-r-2)} \overset{2\ell-r-2}{-}\ d^j_{r(2\ell-r-2)} \overset{2\ell-r-1}{-}\ x^{ji'}_r)$ is a rainbow path between u and $x^{ji'}_r$ in G' (see item 3, and 5-7 of Definiton 9). For $z_{ji'}$, where $i' \in [t] \setminus \{j\}$ the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{-} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-}\ d^j_{(\ell-1)(\ell-1)} \overset{\ell}{-} z_{ji'})$ is a rainbow path between u and $z_{ji'}$ in G' (see item 1, 3, and 5-7 of Definiton 9).

- Consider u^i_{rp} where $r \in [\ell-1]$ and $p \in [k]$. For u^j_{sq} where $s \in [\ell-1]$ and $q \in [k]$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{p+1}{-} u^i_{(r+2)(p+2)} \cdots\ u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-}\ d^j_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-}\ d^j_{(\ell-2)(p+\ell+1-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-2)} \overset{p+2\ell-r-s-2}{-}\ d^j_{s(p+2\ell-r-s-1)} \overset{p+2\ell-r-s-1}{-}\ u^j_{sq})$ is a rainbow path between u^i_{rp} and u^j_{sq} in G' (see item 4, 6, and 7 of Definition 9). For d^j_{sq} where $s \in [\ell-1]$ and $q \in [k]$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{p+1}{-} u^i_{(r+2)(p+2)} \cdots u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-}\ d^j_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-}\ d^j_{(\ell-2)(p+\ell+1-r)} \cdots\ d^j_{(s+1)(p+2\ell-r-s-2)} \overset{p+2\ell-r-s-2}{-} d^j_{sq})$ is a rainbow path between u^i_{rp} and d^j_{sq} in G' (see item 4, 6, and 7 of Definition 9). For $x^{ji'}_s$, where $i' \in [t] \setminus \{j\}$ and $s \in [\ell-2]$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{r+1}{-} u^i_{(r+2)(p+2)} \cdots\ u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-}\ d^j_{(\ell-1)(p+\ell-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-2)} \overset{p+2\ell-r-s-2}{-} d^j_{s(p+2\ell-r-s-2)} \overset{p+2\ell-r-s-1}{-}\ x^{ji'}_s)$ is a rainbow path between u^i_{rq} and $x^{ji'}_s$ in G' (see item 5 to 7 of Definition 9). For $z_{ji'}$, where $i' \in [t] \setminus \{j\}$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{r+1}{-} u^i_{(r+2)(p+2)} \cdots u^i_{(\ell-2)(p+\ell-2-r)} \overset{p+\ell-2-r}{-} u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-}\ d^j_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-r}{-} z_{ji'}$ is a rainbow pat between u^i_{rp} and $z_{ji'}$ in G' (see item 4, 6, and 7 of Definition 9).
- Consider d^i_{rp} where $r \in [\ell-1]$ and $p \in [k]$. For d^j_{sq}, where $s \in [\ell-1]$ and $q \in [k]$ the path $(d^i_{rp} \overset{p}{-} u^i_{r(p+1)} \overset{p+1}{-} u^i_{(r+1)(p+2)} \overset{p+2}{-} u^i_{(r+2)(p+3)} \cdots\ u^i_{(\ell-2)(p+\ell-1-r)} \overset{p+\ell-1-r}{-} u^i_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-} d^j_{(\ell-1)(p+\ell+1-r)} \overset{p+\ell+1-r}{-}\ d^j_{(\ell-2)(p+\ell+2-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-1)} \overset{p+2\ell-r-s-1}{-} d^j_{sq})$ is a rainbow path between d^i_{rp} and d^j_{sq} in G' (see item 4, 6 and 7 of Definition 9). For $x^{ji'}_s$ where $i' \in [t] \setminus \{j\}$ and $s \in [\ell-2]$ the path $(d^i_{rp} \overset{p}{-} u^i_{r(p+1)} \overset{p+1}{-} u^i_{(r+1)(p+2)} \overset{p+2}{-} u^i_{(r+2)(p+3)} \cdots\ u^i_{(\ell-2)(p+\ell-1-r)} \overset{p+\ell-1-r}{-} u^i_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-}\ d^j_{(\ell-1)(p+\ell+1-r)} \overset{p+\ell+1-r}{-}\ d^j_{(\ell-2)(p+\ell+2-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-1)} \overset{p+2\ell-r-s-1}{-}\ d^j_{s(p+2\ell-r-s-1)} \overset{p+2\ell-r-s}{-} x^{ji'}_s)$ is a rainbow path between d^i_{rp} and $x^{ji'}_s$ in G' (see item 4-7 of Definition 9). For $z_{ji'}$ where $i' \in [t] \setminus \{j\}$ the path $(d^i_{rp} \overset{p}{-}\ u^i_{r(p+1)} \overset{p+1}{-}\ u^i_{(r+1)(p+2)} \overset{p+2}{-} u^i_{(r+2)(p+3)} \cdots\ u^i_{(\ell-2)(p+\ell-1-r)} \overset{p+\ell-1-r}{-} u^i_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-}\ d^j_{(\ell-1)(p+\ell-r)} \overset{p+\ell+1-r}{-}\ z_{ji'}$ is a rainbow path between d^i_{rp} and $z_{ji'}$ in G' (see item 4, 6 and 7 of Definition 9).
- For x^{ij}_r and $x^{ji'}_s$ where $i' \in [t] \setminus \{j\}$ and $r, s \in [k]$ the path $x^{ij}_r \overset{r-1}{-} d^i_{r(r-2)} \overset{r-2}{-} u^i_{rr} \overset{r}{-} u^i_{(r+1)(r+1)} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} d^j_{(\ell-1)\ell} \overset{\ell}{-} d^j_{(\ell-2)(\ell+1)} \cdots d^j_{(s-1)(2\ell-s-2)} \overset{2\ell-s-2}{-} d^j_{s(2\ell-s-2)} \overset{2\ell-s-1}{-} x^{ji'}_s$ is a rainbow path between x^{ij}_r and $x^{ji'}_s$ in G' (see item 4 to 7 of Definition 9). For x^{ij}_r and $z_{ji'}$ where $i' \in [t] \setminus \{j\}$ and $r \in [\ell-1]$ the path $x^{ij}_r \overset{r-1}{-} d^i_{r(r-2)} \overset{r-2}{-} u^i_{rr} \overset{r}{-} u^i_{(r+1)(r+1)} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} d^j_{(\ell-1)(\ell-1)} \overset{\ell}{-} z_{ji'}$ is a rainbow path between x^{ij}_r and $z_{ji'}$ in G' (see item 4 to 7 of Definition 9).

◀

We now establish equivalence between the instance G of k-Coloring and the instance G' of Rainbow k-Coloring.

▶ **Lemma 14.** *G' is a* yes *instance of k-*Coloring *if and only if G' is a* yes *instance of* Rainbow *k-*Coloring.

Proof. In the forward direction, let G be a *yes* instance of k-Coloring, and $c : V(G) \to [k]$ be one of its solution. Let $c_R : E(G') \to [k]$ be the coloring given by Definition 9 with the given coloring c of G. From Lemma 12 and 13 it follows that c_R is a solution to Rainbow k-Coloring in G'.

In the reverse direction, let G' be a *yes* instance of Rainbow k-Coloring, and $c_R : E(G') \to [k]$ be one of its solution. We create coloring $c : V(G) \to [k]$ as follows. For $i \in [t]$ and $v \in C_i$, we let $c(v) = c_R(c_i, v)$. We show that c is a valid solution to k-Coloring in G. Consider $(u, v) \in E(G)$, where $u \in C_i$ and $v \in C_j$. Note that we have $i \neq j$. Let P be a rainbow path between c_i and c_j in G'. Note that P can have at most k edges. By Lemma 11 we know that $P = (c_i, u, x_1^{ij}, \cdots, x_{\ell-2}^{ij}, z_{ij}, x_{\ell-2}^{ji}, \cdots, x_1^{ji}, v, c_j)$ therefore, by construction of c, we have that $c_R(c_i, u) = c(u) \neq c(v) = c_R(c_i, v)$. This concludes the proof. ◀

▶ **Theorem 15.** Rainbow *k-*Coloring *does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, unless* ETH *fails. Here, n is the number of vertices in the input graph, and k is an even number greater than* 3.

Proof. Follows from construction of an instance G' of Rainbow k-Coloring with $|E(G')| \in \mathcal{O}(|V(G)|)$ for a given instance G of k-Coloring with maximum degree bounded by $2(k-1)$, Lemma 14, and existence of no algorithm running in time $2^{o(n)}n^{\mathcal{O}(1)}$ for k-Coloring on graphs of maximum degree $2(k-1)$ (assuming ETH). ◀

4.3 Lower Bound for Rainbow k-Coloring, $k > 3$ and odd

In this section, we show that Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, for every odd k where $k > 3$. Here, n is the number of vertices in the input graph.

Let G be an instance of k-Coloring on n vertices with maximum degree bounded by $2(k-1)$. Here, $k > 3$ and k is an odd number. We start by computing (in polynomial time) a harmonious coloring φ of G with $t \in \mathcal{O}(\sqrt{n})$ color classes such that each color class contains at most $\mathcal{O}(\sqrt{n})$ vertices [11, 15, 24, 28]. Let $C_1, C_2, \cdots, C_t$ be the color classes of φ. From the discussion in Section 3, we assume that for $i, j \in [t]$, $i \neq j$ there is exactly one edge between C_i and C_j in G. As discussed in Section 4.2, we modify the graph G and its harmonious coloring φ, to obtain a more structured (equivalent) instance of k-Coloring. This is achieved by adding k new vertices $v_{i1}^*, v_{i2}^*, \cdots, v_{ik}^*$ to C_i (and G) for each $i \in [t]$. The purpose of adding these k new vertices is to ensure that if G is a *yes* instance of k-Coloring then there is a k-coloring c of G, such that for each $i \in [t]$ and $j \in [k]$, we have $c^{-1}(j) \cap C_i \neq \emptyset$. Hereafter, whenever we talk about a solution c to k-Coloring in G (if it exists) we will assume (without explicitly mentioning) that for all $i \in [t]$ and $p \in [k]$ we have $C_i \cap c^{-1}(p) \neq \emptyset$. We now move to description of the reduction.

We first describe the color class gadget $\mathcal{C}_i$, corresponding to each color class C_i, where $i \in [t]$, and gadgets to encode edges in G. We also have a link vertex which is connected to all color class gadgets (but not all vertices). After this, we state connections between color class gadgets and edge gadgets. We let $k = 2\ell + 1$, where $\ell \in \mathbb{N}$ and $\ell \geq 2$. We create an instance G' of Rainbow k-Coloring as described below (see Figure 3).

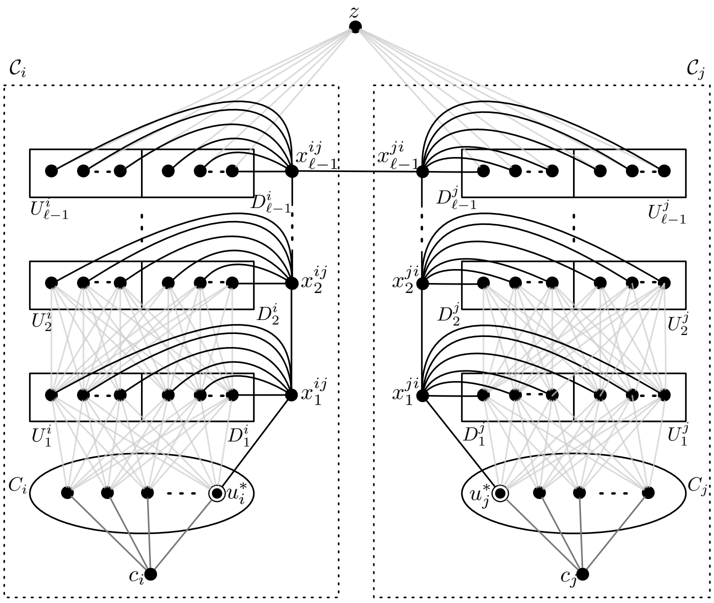

Figure 3 An illustration of (partial) construction the instance G' of k-Coloring, where $k > 3$ and k is odd.

- *Color class gadget.* Consider $i \in [t]$. The color class gadget $\mathcal{C}_i$ comprises of the set C_i, a vertex c_i, and a clique sequence $\mathbb{Z}_i = (U^i_1 \cup D^i_1, \cdots, U^i_{\ell-1} \cup D^i_{\ell-1})$ of order $(2k, \ell-1)$. Here, for each $i \in [\ell-1]$ we have $|U_i| = |D_i| = k$. For $r \in [\ell-1]$ we let $U^i_r = \{u^i_{rp} \mid p \in [k]\}$ and $D^i_r = \{d^i_{rp} \mid p \in [k]\}$. We add all the edges in $\{(c_i, v) \mid v \in C_i\}$ to $E(\mathcal{C}_i)$. Also, we add all the edges in $\{(v, w) \mid v \in C_i, w \in U^i_1 \cup D^i_1\}$ to $E(\mathcal{C}_i)$.
- *Link vertex and its connection to color class gadgets.* We add a vertex z to G'. For each $i \in [t]$, we add all the edges in $\{(z, w) \mid w \in U^i_{\ell-1} \cup D^i_{\ell-1}\}$ to $E(G')$.
- *Edge gadget.* Consider $i, j \in [t]$ with $i \neq j$. Recall that there is exactly one edge between C_i and C_j. Corresponding to this edge we create a path $P = (x^{ij}_1, \cdots, x^{ij}_{\ell-1}, x^{ji}_{\ell-1}, \cdots, x^{ji}_1)$ on $k-3$ vertices, and add it to G'.
- *Connection between color class gadgets and edge gadgets.* Consider $i, j \in [t]$, where $i \neq j$. Let (u^*_i, v^*_j) be the unique edge between C_i and C_j with $u^*_i \in C_i$ and $v^*_j \in C_i$. We add the edges $(u^*_i, x^{ij}_1), (x^{ji}_1, v^*_j)$ to $E(G')$. For each $r \in [\ell-1]$ we add all the edges in $\{(x^{ij}_r, w) \mid w \in U^i_r \cup D^i_r\}$ to $E(G')$. Similarly, we add all the edges in $\{(x^{ji}_r, w) \mid w \in U^j_r \cup D^j_r\}$ to $E(G')$.

This finishes the construction of the instance G' of Rainbow k-Coloring for the given instance G of k-Coloring. Before moving on to proving the equivalence between these instances, we create an edge coloring function $c_R : E(G') \to [k]$. Here, we create c_R based on a solution c to k-Coloring in G, assuming that is G a *yes* instance of k-Coloring. We will follow computation modulo k (color 0 is same as color k).

▶ **Definition 16.** Given a solution c to k-Coloring in G, we construct $c_R : E(G') \to [k]$ as follows.

1. For $i \in [t]$, and $v \in C_i$ we set $c_R(v, c_i) = c(v)$.
2. For $i, j \in [t]$, $i \neq j$ let (u^*_i, v^*_j) be the unique edge between C_i and C_j. Consider the path $P = (u^*_i, x^{ij}_1, \cdots x^{ij}_{\ell-1}, x^{ji}_{\ell-1}, \cdots x^{ji}_1, v^*_j)$. We arbitrarily assign unique integers in

$[k] \setminus \{c(u_i^*), c(v_j^*)\}$ to $c_R(e)$, for each $e \in E(P)$.

3. For $i \in [t]$, a vertex $v \in C_i \cup \{x_1^{ij} \mid j \in [t] \setminus \{i\}\}$, and $p \in [k]$ we set $c_R(v, u_{1p}^i) = p - 1$, and $c_R(v, d_{1p}^i) = p$.
4. For $i \in [t]$, $r \in [\ell - 1]$, and $p, q \in [k]$ we set $c_R(d_{rp}^i, u_{rq}^i) = p$.
5. For $i, j \in [t]$, where $i \neq j$, $r \in [\ell - 1]$, and $p \in [k]$ we set $c_R(x_r^{ij}, u_{rp}^i) = p$, and $c_R(x_r^{ij}, d_{rp}^i) = p + 1$.
6. For $i \in [t]$, $r \in [\ell - 2]$, $p, q \in [k]$ we set $c_R(d_{(r+1)p}^i, d_{rq}^i) = p$, and $c_R(u_{rp}^i, u_{(r+1)q}^i) = p$.
7. For $i \in [t]$, $p \in [k]$ we set $c_R(u_{(\ell-1)p}^i, z) = p$, and $c_R(d_{(\ell-1)p}^i, z) = p - 1$.
8. For $i \in [t]$, $r \in [\ell - 2]$, $p, q \in [k]$ we set $c_R(u_{rp}^i, d_{(r+1)q}^i) = q$ and $c_R(u_{(r+1)p}^i, d_{rq}^i) = p$.
9. For all $i \in [t]$, $r \in [\ell]$, $p, q \in [k]$, where $p \neq q$ we set $c_R(u_{rp}^i, u_{rq}^i) = k$.
10. For all the remaining edges in $E(G')$, c_R assigns it an integer in $[k]$ arbitrarily.

For a vertex $v \in V(G')$, by T_v we denote the breadth first search tree in G' with v as the root vertex. We let $L_0^v = \{v\}$. For $i \in [n]$, by L_i^v we denote the set vertices which are at a distance i from v in T_v. Here, the distance between $u \in V(G')$ and v denotes the number of edges in the unique path between v and u in T_v.

Next, we prove some lemmata that will be useful in establishing the equivalence between the instance G of k-Coloring and the instance G' of Rainbow k-Coloring.

▶ **Lemma 17.** *For $i, j \in [t]$, where $i \neq j$, let P be a path between c_i and c_j with at most k edges in G'. Then $(x_{\ell-1}^{ij}, x_{\ell-1}^{ij}) \in E(P)$.*

Proof. Consider $i, j \in [t]$, where $i \neq j$. Let P be a path between c_i and c_j with at most k edges in G'. Recall that $N(c_i) = C_i$ and $N(c_j) = C_j$, where $C_i \cap C_j = \emptyset$. Therefore, P must contain an edge (c_i, u) and (v, c_j), where $u \in C_i$ and $v \in C_j$ $(u \neq v)$. We consider the breadth first search trees T_{c_i} and T_{c_j}. We start by looking at first ℓ levels (including level 0). Notice that for $r \in [\ell - 1]$ we have $L_{r+1}^{c_i} = U_r^i \cup D_r^i \cup \{x_r^{ij'} \mid j' \in [t] \setminus \{i\}\}$, and $L_1^{c_i} = C_i$. Similarly, for $r \in [\ell - 1]$ we have $L_{r+1}^{c_j} = U_r^j \cup D_r^j \cup \{x_r^{ji'} \mid i' \in [t] \setminus \{j\}\}$, and $L_1^{c_j} = C_j$. For all $r, r' \in [\ell]$ we have $L_r^{c_i} \cap L_{r'}^{c_j} = \emptyset$. Therefore, for each $w \in \{c_i, c_j\}$ and $r \in [\ell]$, P must contain a vertex from L_r^w. This implies that P must contain at least $2\ell + 2 = k + 1$ vertices. Since P is a path on at most k edges between c_i and c_j, P must contain exactly one vertex from each L_r^w, where $w \in \{c_i, c_j\}$ and $r \in [\ell - 1]$. Moreover, the vertices $u_{\ell-1}^* \in L_\ell^{c_i} \cap V(P)$ and $v_{\ell-1}^* \in L_\ell^{c_j} \cap V(P)$ must contain an edge between them. By construction of G', there is exactly one edge, namely $(x_{\ell-1}^{ij}, x_{\ell-1}^{ji})$ between a vertex in $L_\ell^{c_i}$ and a vertex in $L_\ell^{c_j}$. Therefore, P must contain the edge $(x_{\ell-1}^{ij}, x_{\ell-1}^{ji})$. ◀

▶ **Lemma 18.** *For $i, j \in [t]$, where $i \neq j$ let (u_i^*, v_j^*) be the unique edge between C_i and C_j with $u_i^* \in C_i$ and $v_j^* \in C_j$. There is exactly one path, namely $(c_i, u_i^*, x_1^{ij}, \cdots, x_{\ell-1}^{ij}, x_{\ell-1}^{ji}, \cdots, x_1^{ji}, v_j^*, c_j)$ in G' between c_i and c_j that has at most k edges.*

Proof. Consider $i, j \in [t]$, where $i \neq j$. Let $u_i^* \in C_i$, $v_j^* \in C_j$ be the vertices such that $(u_i^*, v_j^*) \in E(G')$. Also, let P be a (simple) path between c_i and c_j with at most k edges in G'. By construction of G', P contains an edge (c_i, u) and an edge (v, c_j), where $u \in C_i$ and $v \in C_j$, respectively. Recall that for $r \in [\ell - 1]$ we have $L_{r+1}^{c_i} = U_r^i \cup D_r^i \cup \{x_r^{ij'} \mid j' \in [t] \setminus \{i\}\}$, $L_{r+1}^{c_j} = U_r^j \cup D_r^j \cup \{x_r^{ji'} \mid i' \in [t] \setminus \{i\}\}$, $L_1^{c_i} = C_i$, and $L_1^{c_j} = C_j$. Since for all $r, r' \in [\ell]$ we have $L_r^{c_i} \cap L_{r'}^{c_j} = \emptyset$. Therefore, for each $w \in \{c_i, c_j\}$ and $r \in [\ell]$, P contains exactly one vertex from L_r^w. From Lemma 17 we know that $(x_{\ell-1}^{ij}, x_{\ell-1}^{ji}) \in E(P)$. Therefore, either P contains a sub-path P_1 from c_i to $x_{\ell-1}^{ji}$ and a sub-path P_2 from $x_{\ell-1}^{ij}$ to c_j or it contains a sub-path P_1' from c_i to $x_{\ell-1}^{ij}$ and a sub-path P_2' from $x_{\ell-1}^{ji}$ to c_j. Consider the case when P contains a sub-path P_1 from c_i to $x_{\ell-1}^{ji}$ and a sub-path P_2 from $x_{\ell-1}^{ij}$ to c_j. Since P is simple path therefore, $E(P_1) \cap E(P_2) = \emptyset$, and $V(P_1) \cap V(P_2) = \emptyset$. Moreover, any path from c_i to $x^{ji}_{\ell-1}$ contains at least $\ell+1$ edges. This is implied from the fact that $x^{ji}_{\ell-1} \in L^{c_i}_{\ell+1}$. Similarly, any path from c_j to $x^{ij}_{\ell-1}$ contains at least $\ell+1$ edges. But then P contains at least $2(\ell+1)+1 > k$ edges.

Next, consider the case when P contains a sub-path P'_1 from c_i to $x^{ij}_{\ell-1}$ and a sub-path P'_2 from $x^{ji}_{\ell-1}$ to c_j. Notice that the shortest path from c_i to $x^{ij}_{\ell-1}$ has at least ℓ edges. This follows from the fact that $x^{ij}_{\ell-1} \in L^{c_i}_{\ell}$. Similarly, the shortest path from $x^{ji}_{\ell-1}$ to c_j has at least ℓ edges. This implies that P'_1 and P'_2 both have exactly ℓ edges. We now show that $P'_1 = (c_i, u^*_i, x^{ij}_1, \cdots, x^{ij}_{\ell-1})$ (and an analogous argument can be applied for P'_2). Consider the smallest number $r \in [\ell-1]$ such that $x^{ij}_{r-1} \notin V(P'_1)$ and $x^{ij}_r \in V(P'_1)$. Here, for $r = 1$ we assume that $x^{ij}_{r-1} = u^*_i$. If such an r does not exists then we have $P'_1 = (c_i, u^*_i, x^{ij}_1, \cdots, x^{ij}_{\ell-1})$. This follows from the fact that $x^{ij}_{\ell-1} \in V(P'_1)$, the unique vertex in C_i that is adjacent to x^{ij}_1 is u^*_i, and $|V(P) \cap C_i| = 1$. We now consider the case when such an r exists. Since for each $r' \in [\ell]$ we have $|V(P'_1) \cap L^{c_i}_{r'}| = 1$ therefore, there exists $x \in L^{c_i}_{r-1} \cap V(P'_1)$. By construction of G' (and r), we have $(x, x^{ij}_r) \notin E(G')$. This together with the fact that for each $r' \in [\ell]$ we have $|V(P'_1) \cap L^{c_i}_{r'}| = 1$ implies that such an r cannot exists. This concludes the proof. ◀

▶ **Lemma 19.** *Let G be a yes instance of k-Coloring, and c be one of its solution. Furthermore, let $c_R : E(G') \to [k]$ be the coloring given by Definition 16 for the coloring c of G. For all $i \in [t]$, and $u, v \in V(\mathcal{C}_i) \cup \{x^{ij}_r \mid j \in [t] \setminus \{i\}, r \in [\ell-1]\} \cup \{z\}$ there is a rainbow path between u and v in G'.*

Proof. Consider $i \in [t]$. Recall that $V(\mathcal{C}_i) = \{c_i\} \cup C_i \cup \{u^i_{rp}, d^i_{rp} \mid r \in [\ell-1], p \in [k]\}$. Let $U_i = \cup_{r \in [\ell-1]} U^i_r$, $D_i = \cup_{r \in [\ell-1]} D^i_r$, and $X_i = \{x^{ij}_r \mid j \in [t] \setminus \{i\}, r \in [\ell-1]\}$. We will argue only for non-adjacent pair of vertices, since we trivially have a rainbow path between pair of vertices that have an edge between them. Therefore, we argue consider pairs of vertices in the following sets.

- $A_1 = \{(c_i, v) \mid v \in U_i \cup D_i \cup X_i \cup \{z\}\}$.
- $A_2 = \{(u, v) \mid u \in C_i, v \in (U_i \setminus U^i_1) \cup (D_i \setminus D^i_1) \cup X_i \cup \{z\}\}$.
- $A_3 = \{(u, v) \mid u \neq v, u \in U_i, v \in U_i \cup D_i \cup X_i \cup \{z\}\}$.
- $A_4 = \{(u, v) \mid u \neq v, u \in D_i, v \in D_i \cup X_i \cup \{z\}\}$.
- $A_5 = \{(u, v) \mid u \neq v, u \in X_i, v \in X_i \cup \{z\}\}$

We now show that each pair in $\cup_{r \in [5]} A_r$ has a rainbow path between them.

- For $p \in [k]$, let $v^*_p \in C_i$ be a vertex such that $c_R(c_i, v^*_p) = p$. For each $x \in X_i$, by construction of c_R (and G') it follows that there is a rainbow path between c_i and x (see item 1 and 2 of Definition 16). The path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{\ell-2(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} z)$ is a rainbow path between c_i and z in G' (see item 1, 3, 6, and 7 of Definition 16). For $p \in [k]$, the path $(c_i \overset{p}{—} v^*_p \overset{p-1}{—} u^i_{1p})$ is a rainbow path between c_i and u^i_{1p} (see item 1 and 3 of Definition 16). Similarly, $(c_i \overset{p+1}{—} v^*_{p+1} \overset{p}{—} d^i_{1p})$ is a rainbow path between c_i and d^i_{1p}. For $r \in [\ell-1] \setminus \{1\}$ and $p \in [k]$ the path $(c_i \overset{k-1}{—} v^*_{k-1} \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(r-1)(r-1)} \overset{r-1}{—} u^i_{rp})$ is a rainbow path between c_i and u^i_{rp} in G' (see item 1, 3, and 6 of Definition 16). Similarly, for $r \in [\ell-1] \setminus \{1\}$, $p \in [k]$ the path $(c_i \overset{p+r}{—} v^*_{p+r} \overset{p+r-1}{—} d^i_{1(p+r-1)} \overset{p+r-2}{—} d^i_{2(p+r-2)} \cdots d^i_{(r-1)(p+2)} \overset{p+1}{—} d^i_{r(p+1)} \overset{p}{—} d^i_{rp})$ is a rainbow path between c_i and d^i_{rp} in G'.
- Consider $v \in C_i$. The path $(v \overset{k}{—} u^i_{11} \overset{1}{—} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{—} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{—} z)$ is a rainbow path between v and z in G' (see item 3, 6, and 7 of Definition 16). For $r \in [\ell-1] \setminus \{1\}$

and $p \in [k]$ the path $(v \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(r-1)(r-1)} \overset{r-1}{-} u^i_{rp})$ is a rainbow path between v and u^i_{rp} in G' (see item 3 and 6 of Definition 16). Similarly, for $r \in [\ell-1] \setminus \{1\}$, $p \in [k]$ the path $(v \overset{p+r-1}{-} d^i_{1(p+r-1)} \overset{p+r-2}{-} d^i_{2(p+r-2)} \cdots d^i_{(r-2)(p+2)} \overset{p+1}{-} d^i_{(r-1)(p+1)} \overset{p}{-} d^i_{rp})$ is a rainbow path between v and d^i_{rp} in G'. For x^{ij}_r, where $j \in [t] \setminus \{i\}$ and $r \in [\ell-1]$ the path $(v \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(r-1)(r-1)} \overset{r-1}{-} u^i_{rr} \overset{r}{-} x^{ij}_r)$ is a rainbow path between v and x^{ij}_r in G' (see item 3, 5, and 6 of Definition 16).

- Consider a vertex u^i_{rp}, where $r \in [\ell-1]$ and $p \in [k]$. Also, consider a vertex u^i_{sq}, where $s \in [\ell-1] \setminus \{r\}$ and $q \in [k]$. Without loss of generality we assume that $r < s$. The path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{p+1}{-} \cdots u^i_{(s-1)(p+s-r-1)} \overset{p+s-r-1}{-} u^i_{sq})$ is a rainbow path between u^i_{rp} and u^i_{sq} in G' (see item 6 of Definition 16). For a vertex d^i_{sq}, where $s \in [\ell-1] \setminus \{r\}$ and $q \in [k]$. If $r < s$ then path $(d^i_{sq} \overset{q}{-} d^i_{(s-1)(q+1)} \cdots d^i_{(r+1)(q+s-r-1)} \overset{q+s-r-1}{-} u^i_{rp})$ is a rainbow path between u^i_{rp} and d^i_{sq} in G' (see item 6 of Definition 16). Otherwise, $r > s$ and the path $(d^i_{sq} \overset{q}{-} u^i_{s(q+1)} \overset{q+1}{-} u^i_{(s+1)(q+2)} \overset{q+2}{-} u^i_{(s+2)(q+3)} \cdots u^i_{(r-1)(q+r-s)} \overset{q+r-s}{-} u^i_{rp})$ is a rainbow path between u^i_{rp} and d^i_{sq} in G' (see item 4 and 6 of Definition 16). For a vertex $x^{ij}_s \in X_i$, where $s \in [\ell-1] \setminus \{r\}$ and $j \in [t] \setminus \{i\}$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{p+1}{-} \cdots u^i_{s(p+s-r)} \overset{p+s-r}{-} x^{ij}_s)$ is a rainbow path between u^i_{rp} and x^{ij}_s in G' (see item 5 and 6 of Definition 16). The path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{p+1}{-} \cdots u^i_{(\ell-1)(p+\ell-r-1)} \overset{p+\ell-r-1}{-} z)$ is a rainbow path between u^i_{rp} and z in G' (see item 6 and 7 of Definition 16).
- Consider a vertex d^i_{rp}, where $r \in [\ell-1]$ and $p \in [k]$. Next, consider a vertex d^i_{sq}, where $s \in [\ell-1] \setminus \{r\}$ and $q \in [k]$. Without loss of generality we assume $r < s$. The path $(d^i_{rp} \overset{q+s-r-1}{-} d^i_{(r+1)(q+s-r-1)} \cdots d^i_{(s-1)(q+1)} \overset{s}{-} d^i_{sq})$ is a rainbow path between d^i_{rp} and d^i_{sq} in G' (see item 6 of Definition 16). The path $(d^i_{rp} \overset{p+1}{-} d^i_{(r+1)(p+1)} \cdots d^i_{(\ell-2)(p+\ell-r-2)} \overset{p+\ell-r-2}{-} d^i_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r-1}{-} z)$ is a rainbow path between d^i_{sq} and z in G' (see item 6 and 7 of Definition 16). For x^{ij}_s, where $s \in [\ell-1] \setminus \{r\}$ and $j \in [t] \setminus \{i\}$ consider the following. If $r < s$ then the path $(d^i_{rp} \overset{p+1}{-} d^i_{(r+1)(p+1)} \overset{p+2}{-} \cdots d^i_{(s-1)(p+s-r-1)} \overset{p+s-r}{-} d^i_{s(p+s-r)} \overset{p+s-r+1}{-} x^{ij}_s)$ is a rainbow path between d^i_{pr} and x^{ij}_s in G' (see item 5 and 6 of Definition 16). Otherwise, $s < r$ and the path $(d^i_{rp} \overset{p}{-} d^i_{(r-1)(p+1)} \overset{p+1}{-} d^i_{(r-2)(p+2)} \cdots d^i_{(s+1)(p+r-s-1)} \overset{p+r-s-1}{-} d^i_{s(p+r-s-1)} \overset{p+r-s}{-} x^{ij}_s)$ is a rainbow path between d^i_{pr} and x^{ij}_s in G' (see item 5 and 6 of Definition 16).
- Consider a vertex x^{ij}_r, where $j \in [t] \setminus \{i\}$ and $r \in [\ell-1]$. For all $x \in \{x^{ij}_s \mid s \in [\ell-1]\}$, by the construction of G' and c_R we have a rainbow path between x^{ij}_r and x^{ij}_s (see item 2 in the Definition 16). For a vertex $x^{ij'}_r$, where $j' \in [t] \setminus \{i,j\}$ by construction of c_R the path $(x^{ij}_r, u^i_{r1}, u^i_{r2}, x^{ij'}_r)$ is a rainbow path between x^{ij}_r and $x^{ij'}_r$ (see item 5 and 9 of Definition 16). Next, consider a vertex $x^{ij'}_s$ where $j' \in [t] \setminus \{i,j\}$ and $s \in [\ell-1] \setminus \{r\}$. Without loss of generality we assume that $r < s$. The path $(x^{ij}_r \overset{r}{-} u^i_{r(r+1)} \overset{r+1}{-} u^i_{(r+1)(r+2)} \cdots u^i_{s(r+1+s-r)} \overset{r+1+s-r}{-} x^{ij'}_s)$ is a rainbow path between x^{ij}_r and x^{ij}_s (see item 5 and 6 of Definition 16). The path $(x^{ij}_r \overset{r}{-} u^i_{r(r+1)} \overset{r+1}{-} u^i_{(r+1)(r+2)} \cdots u^i_{(\ell-1)\ell} \overset{\ell}{-} z)$ is a rainbow path between x^{ij}_r and z (see item 5, 6, and 7 of Definition 16).

◀

▶ **Lemma 20.** *Let G be a yes instance of k-*COLORING*, and c be one of its solution. Furthermore, let $c_R : E(G') \to [k]$ be the coloring given by Definition 16 for the coloring c of G. For all $i, j \in [t]$ where $i \neq j$, $u \in V(\mathcal{C}_i) \cup \{x^{ij'}_r \mid j' \in [t] \setminus \{i\}, r \in [\ell-1]\}$ and $v \in \mathcal{C}_j \cup \{x^{ji'}_r \mid i' \in [t] \setminus \{j\}, r \in [\ell-1]\}$ there is a rainbow path between u and v in G'.*

Proof. For $i \in [t]$, let $U_i = \cup_{r \in [\ell-1]} U^i_r$, $D_i = \cup_{r \in [\ell-1]} D^i_r$, and $X_i = \{x^{ij}_r \mid j \in [t] \setminus \{i\}, r \in [\ell-1]\}$. For $i, j \in [t]$, where $i \neq j$ we consider the pairs in the following sets.

- $A_1 = \{(c_i, v) \mid v \in \{c_j\} \cup C_j \cup U_j \cup D_j \cup X_j\}$.
- $A_2 = \{(u, v) \mid u \in C_i, v \in C_j \cup U_j \cup D_j \cup X_j\}$.
- $A_3 = \{(u, v) \mid u \in U_i, v \in U_j \cup D_j \cup X_j\}$.
- $A_3 = \{(u, v) \mid u \in D_i, v \in D_j \cup X_j\}$.
- $A_3 = \{(u, v) \mid u \in X_i, v \in X_j\}$.

Although, $\cup_{r\in[5]} A_r$ does not contain all the pairs in $(V(\mathcal{C}_i) \cup X_i) \times V(\mathcal{C}_j \cup X_j)$, but it is enough to argue about pairs of vertices in $\cup_{r\in[5]} A_r$. This follows from the fact that for all the missing pairs in $\cup_{r\in[5]} A_r$, we can obtain rainbow path by a symmetric argument (swapping roles of i and j). Next, we proceed to prove that we have a rainbow path between every pair of vertices in $\cup_{r\in[5]} A_r$.

- Recall that by construction of c_R (and G') we have a rainbow path between c_i and c_j (see item 1 and 2 of Definition 16). For $v \in C_j$ the path $(c_i \overset{k-1}{-} v^*_{k-1} \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{2(2\ell-2)} \overset{2\ell-2}{-} d^j_{1(2\ell-1)} \overset{2\ell-1}{-} v)$ is a rainbow path between c_i and v in G' (see item 1, 3, 6, and 7 of Definition 16). For u^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(c_i \overset{k-1}{-} v^*_{k-1} \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{-} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(r+1)(2\ell-r-1)} \overset{2\ell-r-1}{-} d^j_{r(2\ell-r)} \overset{2\ell-r}{-} u^j_{rp})$ is a rainbow path between c_i and u^j_{rp} in G' (see item 1, 3, 4, 6 and 7 of Definition 16). For d^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(c_i \overset{k-1}{-} v^*_{k-1} \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{-} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(r+1)(2\ell-r-1)} \overset{2\ell-r-1}{-} d^j_{rp})$ is a rainbow path between c_i and d^j_{rp} in G' (see item 1, 3, 6, and 7 of Definition 16). For x^{ji}_r, where $r \in [\ell-1]$ by construction of c_R and G' we have rainbow path between c_i and x^{ji} (see item 1 and 2 of Definition 16). For $x^{ji'}_r$, where $i' \in [t] \setminus \{i, j\}$ and $r \in [\ell-1]$ consider the path $(c_i \overset{k-1}{-} v^*_{k-1} \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{-} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(r+1)(2\ell-r-1)} \overset{2\ell-r-1}{-} d^j_{r(2\ell-r-1)} \overset{2\ell-r}{-} x^{ji'}_r)$ is a rainbow path between c_i and $x^{ji'}_r$ in G' (see item 1, 3, and 5-7 of Definiton 16).
- Consider $u \in C_i$. For $v \in C_j$ the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{-} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{2(2\ell-2)} \overset{2\ell-2}{-} d^j_{1(2\ell-1)} \overset{2\ell-1}{-} v)$ is a rainbow path between u and v in G' (see item 3, 6, and 7 of Definition 16). For u^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-2)(\ell-2)} \overset{\ell-2}{-} u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(r+1)(2\ell-r-1)} \overset{2\ell-r-1}{-} d^j_{r(2\ell-r)} \overset{2\ell-r}{-} u^j_{rp})$ is a rainbow path between u and u^j_{rp} in G' (see item 3, 4, 6, and 7 of Definition 16). For d^j_{rp}, where $r \in [\ell-1]$ and $p \in [k]$ the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(r+1)(2\ell-r-1)} \overset{2\ell-r-1}{-} d^j_{rp})$ is a rainbow path between u and d^j_{rp} in G' (see item 3, 6, and 7 of Definition 16). For $x^{ji'}_r$, where $i' \in [t] \setminus \{j\}$ and $r \in [\ell-1]$ consider the path $(u \overset{k}{-} u^i_{11} \overset{1}{-} u^i_{22} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(r+1)(2\ell-r-1)} \overset{2\ell-r-1}{-} d^j_{r(2\ell-r-1)} \overset{2\ell-r}{-} x^{ji'}_r)$ is a rainbow path between u and $x^{ji'}_r$ in G' (see item 3, 5, 6, and 7 of Definiton 16).
- Consider u^i_{rp} where $r \in [\ell-1]$ and $p \in [k]$. For u^j_{sq} where $s \in [\ell-1]$ and $q \in [k]$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{r+1}{-} u^i_{(r+2)(p+2)} \cdots u^i_{(\ell-2)(p+\ell-2-r)} \overset{p+\ell-2-r}{-} u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-} z \overset{p+\ell-r}{-} d^j_{(\ell-1)(p+\ell+1-r)} \overset{p+\ell+1-r}{-} d^j_{(\ell-2)(p+\ell+2-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-1)} \overset{p+2\ell-r-s-1}{-} d^j_{s(p+2\ell-r-s)} \overset{p+2\ell-r-s}{-} u^j_{sq})$ is a rainbow path between u^i_{rp} and u^j_{sq} in G' (see item 4, 6, and 7 of Definition 16). For d^j_{sq}, where $s \in [\ell-1]$ and $q \in [k]$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{r+1}{-} u^i_{(r+2)(p+2)} \cdots u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-} z \overset{p+\ell-r}{-} d^j_{(\ell-1)(p+\ell+1-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-1)} \overset{p+2\ell-r-s-1}{-} d^j_{sq})$ is a

rainbow path between u^i_{rp} and d^j_{sq} in G' (see item 4, 6 and 7 of Definition 16). For $x^{ji'}_s$, where $i' \in [t] \setminus \{j\}$ and $s \in [\ell-1]$ the path $(u^i_{rp} \overset{p}{-} u^i_{(r+1)(p+1)} \overset{r+1}{-} u^i_{(r+2)(p+2)} \cdots u^i_{(\ell-1)(p+\ell-1-r)} \overset{p+\ell-1-r}{-} z \overset{p+\ell-r}{-} d^j_{(\ell-1)(p+\ell+1-r)} \cdots d^j_{(s+1)(p+2\ell-r-s-1)} \overset{p+2\ell-r-s-1}{-} d^j_{s(p+2\ell-r-s-1)} \overset{p+2\ell-r-s}{-} x^{ji'}_s)$ is a rainbow path between u^i_{rq} and $x^{ji'}_s$ in G' (see item 4 to 7 of Definition 16).

- Consider a vertex d^i_{rp}, where $r \in [\ell-1]$ and $p \in [k]$. For d^j_{sq}, where $s \in [\ell-1]$ and $q \in [k]$ the path $(d^i_{rp} \overset{p}{-} u^i_{r(p+1)} \overset{p+1}{-} u^i_{(r+1)(p+2)} \overset{p+2}{-} u^i_{(r+2)(p+3)} \cdots u^i_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-} z \overset{p+\ell+1-r}{-} d^j_{(\ell-1)(p+\ell+2-r)} \overset{p+\ell+2-r}{-} d^j_{(\ell-2)(p+\ell+3-r)} \cdots d^j_{(s+1)(p+2\ell-r-s)} \overset{p+2\ell-r-s}{-} d^j_{sq})$ is a rainbow path between d^i_{rp} and u^j_{sq} in G' (see item 4, 6, and 7 of Definition 16). For $x^{ji'}_s$ where $i' \in [t] \setminus \{j\}$ and $s \in [\ell-1]$ the path $(d^i_{rp} \overset{p}{-} u^i_{r(p+1)} \overset{p+1}{-} u^i_{(r+1)(p+2)} \overset{p+2}{-} u^i_{(r+2)(p+3)} \cdots u^i_{(\ell-1)(p+\ell-r)} \overset{p+\ell-r}{-} z \overset{p+\ell+1-r}{-} d^j_{(\ell-1)(p+\ell+2-r)} \overset{p+\ell+2-r}{-} d^j_{(\ell-2)(p+\ell+3-r)} \cdots d^j_{(s+1)(p+2\ell-r-s)} \overset{p+2\ell-r-s}{-} d^j_{s(p+2\ell-r-s)} \overset{p+2\ell-r-s+1}{-} x^{ji'}_s)$ is a rainbow path between d^i_{rp} and $x^{ji'}_s$ in G' (see item 4, 6, and 7 of Definition 16).
- For x^{ij}_r and $x^{ji'}_s$ where $i' \in [t] \setminus \{j\}$ and $r, s \in [\ell-1]$ the path $x^{ij}_r \overset{r-1}{-} d^i_{r(r-2)} \overset{r-2}{-} u^i_{rr} \overset{r}{-} u^i_{(r+1)(r+1)} \cdots u^i_{(\ell-1)(\ell-1)} \overset{\ell-1}{-} z \overset{\ell}{-} d^j_{(\ell-1)(\ell+1)} \overset{\ell+1}{-} d^j_{(\ell-2)(\ell+2)} \cdots d^j_{(s+1)(2\ell-s-1)} \overset{2\ell-s-1}{-} d^j_{s(2\ell-s-1)} \overset{2\ell-s}{-} x^{ji'}_s$ is a rainbow path between x^{ij}_r and $x^{ji'}_s$ in G' (see item 4 to 7 of Definition 16).

◀

We now establish equivalence between the instance G of Rainbow k-Coloring and the instance G' of Rainbow 3-Coloring.

▶ **Lemma 21.** *G' is a* yes *instance of k-Coloring if and only if G' is a* yes *instance of Rainbow k-Coloring.*

Proof. In the forward direction, let G be a *yes* instance of k-Coloring, and $c : V(G) \to [k]$ be one of its solution. Let $c_R : E(G') \to [k]$ be the coloring given by Definition 16 with the given coloring c of G. From Lemma 19 and 20 it follows that c_R is a solution to Rainbow k-Coloring in G'.

In the reverse direction, let G' be a *yes* instance of Rainbow k-Coloring, and $c_R : E(G') \to [k]$ be one of its solution. We create coloring $c : V(G) \to [k]$ as follows. For $i \in [t]$ and $v \in C_i$, we let $c(v) = c_R(c_i, v)$. We show that c is a valid solution to k-Coloring in G. Consider $(u, v) \in E(G)$, where $u \in C_i$ and $v \in C_j$. Note that we have $i \neq j$. Let P be a rainbow path between c_i and c_j in G'. Observe that P can have at most k edges. By Lemma 18 we know that $P = (c_i, u, x^{ij}_1, \cdots, x^{ij}_{\ell-1}, x^{ji}_{\ell-1}, \cdots, x^{ji}_1, v, c_j)$ therefore, by construction of c, we have that $c_R(c_i, u) = c(u) \neq c(v) = c_R(c_i, v)$. This concludes the proof. ◀

▶ **Theorem 22.** Rainbow k-Coloring *does not admit an algorithm running in time $2^{o(|E(G)|)} n^{\mathcal{O}(1)}$, unless* ETH *fails. Here, n is the number of vertices in the input graph, and k is an odd number greater than* 3.

Proof. Follows from construction of an instance G' of Rainbow k-Coloring with $|E(G')| \in \mathcal{O}(|V(G)|)$ for a given instance G of k-Coloring with maximum degree bounded by $2(k-1)$, Lemma 21, and existence of no algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ for k-Coloring on graphs of maximum degree $2(k-1)$ (assuming ETH). ◀

5 FPT Algorithm for Subset Rainbow k-Coloring

In this section, we design an FPT algorithm running in time $\mathcal{O}(2^{|S|}n^{\mathcal{O}(1)})$ for Subset Rainbow k-Coloring, when parameterized by $|S|$. Our algorithm is based on the technique of color coding, which was first introduced by Alon et al. [2]. We first describe a randomized algorithm for Subset Rainbow k-Coloring, which we derandomize using splitters.

The intuition behind the algorithm is as follows. Let (G, S) be an instance of Subset Rainbow k-Coloring on n vertices and m edges. For a solution $c_R : E(G) \to [k]$, to Subset Rainbow k-Coloring in (G, S) the following holds. For each $(u, v) \in S$, there exist a path P from u to v in G with at most k edges such that for all $e, e' \in E(P)$, where $e \neq e'$ we have $c_R(e) \neq c_R(e')$. Therefore, at most $k|S|$ edges in G seems to be "important" for us, *i.e.* if we color at most $k|S|$ edges "nicely" then we would obtain the desired soultion. To capture this, we start by randomly coloring edges in G, hoping that with sufficiently high probability we obtain a coloring that colors the desired set of edges "nicely". Once we have obtained such a "nice" coloring, we employ the algorithm of Kowalik and Lauri [21] to check if there is a rainbow path for each $(u, v) \in S$. We note that we use the algorithm given by [21] instead of the one in [31] because the latter requires exponential space.

Algorithm Rand-SRC. Let $c : E(G) \to [k]$ be a coloring of $E(G)$, where each edge is colored with one of the colors in $[k]$ uniformly and independently at random. If for each $(u, v) \in S$, there is rainbow path between u and v in G' with edge coloring c then the algorithm return c as a solution to Subset Rainbow k-Coloring in (G, S). Otherwise, it returns *no*. We note that for a given graph G with edge coloring c, and vertices u and v, in time $2^k n^{\mathcal{O}(1)}$ time we can check if there is a rainbow path between u and v in G' by using the algorithm given by Corollary 5 in [21]. This completes the description of the algorithm.

We now proceed to show how we can obtain an algorithm with constant success probability.

▶ **Theorem 23.** *There is an algorithm that, given an instance (G, S) of* Subset Rainbow k-Coloring*, in time $2^{\mathcal{O}(|S|k \log k)}n^{\mathcal{O}(1)}$ either returns no or outputs a solution to* Subset Rainbow k-Coloring *in (G, S). Moreover, if the input is a yes instance of* Subset Rainbow k-Coloring*, then it returns a solution with constant probability.*

Proof. We start by showing that Rand-SRC runs in time $2^k n^{\mathcal{O}(1)}$, and given a *yes* instance of Subset Rainbow k-Coloring, outputs a solution with probability at least $2^{-\mathcal{O}(|S|k \log k)}$. Clearly, by repeating Rand-SRC $2^{\mathcal{O}(|S|k \log k)}$ times, we obtain the desired success probability and running time.

The algorithm Rand-SRC starts by coloring edges in G' uniformly and independently at random to obtain a coloring $c : E(G') \to [k]$. This step can be executed in time $\mathcal{O}(m)$. Then, for each pair $(u, v) \in S$, in time $2^k n^{\mathcal{O}(1)}$ it checks if there is a rainbow path between u and v in G for the edge coloring c. If for every pair in S it find a rainbow path between them, it correctly outputs a solution. The correctness and the running time bound of this step relies on the correctness of Corollary 5 of [21]. Otherwise, Rand-SRC outputs *no*. Therefore, we have the desired running time bound.

Towards proving the desired success probability, assume that (G, S) is a *yes* instance of Subset Rainbow k-Coloring, and c_R be one of its solution. Moreover, for a pair $(u, v) \in S$ let P_{uv} be a rainbow path in G. Here, if there are many such paths then we arbitrarily choose one of them. Note that for each $(u, v) \in S$ we have $|E(P)| \leq k$. Consider the set $E_R = \cup_{(u,v)\in S} E(P_{uv})$. We now show that the probability with which $c|_{E_R} = c_R|_{E_R}$ is at least $2^{-\mathcal{O}(|S|k \log k)}$. Notice that there are $k^{|E(G)|}$ many distinct colourings of edges in

G. Moreover, at least $k^{|E(G)|-k|S|}$ of these colorings satisfy the desired property (agree with c_R on edges in E_R). Thus, we obtain the desired success probability bound. ◀

We start by defining some terminologies which will be useful in derandomization of our algorithm (see [12, 29]). An (n, p, ℓ)-*splitter* $\mathcal{F}$, is a family of functions from $[n]$ to ℓ such that for every $S \subseteq [n]$ of size at most p there is a function $f \in \mathcal{F}$ such that f splits S evenly. That is, for all $i, j \in [\ell]$, $|f^{-1}(i)|$ and $|f^{-1}(j)|$ differs by at most 1. Observe that when $\ell \geq p$ then for any $S \subseteq [n]$ of size at most p and a function $f \in \mathcal{F}$ that splits S, we have $|f^{-1}(i) \cap S| \leq 1$, for all $i \in [\ell]$. An (n, ℓ, ℓ)-splitter is called as an (n, ℓ)-*perfect hash family*. Moreover, for any $\ell \geq 1$, we can construct an (n, ℓ)-perfect hash family of size $e^{\ell}\ell^{\mathcal{O}(\log \ell)} \log n$ in time $e^{\ell}\ell^{\mathcal{O}(\log \ell)} n \log n$ [29].

We next move to the description of derandomization of the algorithm presented in Theorem 23. For the sake of simplicity in explanation, we associate each $e \in E(G)$ with a unique integer, say i_e in $[m]$, and whenever we refer to e as an integer, we actually refer to the integer i_e. We start by computing an $(m, k|S|)$-perfect hash family $\mathcal{F}$ of size $e^{k|S|}(k|S|)^{\mathcal{O}(\log k|S|)} \log m$ in time $e^{k|S|}(k|S|)^{\mathcal{O}(\log k|S|)} m \log m$ using the algorithm of Naor et al. in [29]. We will create a family of function $\mathcal{F}'$ from $[m]$ to $[k]$ of size $e^{k|S|}(k|S|)^{\mathcal{O}(\log k|S|)} k^{k|S|} \log m$. Towards this, consider an $f \in \mathcal{F}$ and a partition $\mathcal{P} = \{P_1, P_2, \cdots P_{k'}\}$ of $[k|S|]$ into k' sets, where $k' \leq k$. We let $f_{\mathcal{P}}$ to be the function obtained from f as follows. For each $i \in [k']$ we have $f_{\mathcal{P}}^{-1}(i) = \cup_{x \in P_i} f^{-1}(x)$. For every such pair f and $\mathcal{P}$, we add the function $f_{\mathcal{P}}$ to the set $\mathcal{F}'$. We will call such an $\mathcal{F}'$ as $(m, k|S|, k)$-*unified perfect hash* family. Observe that $\mathcal{F}'$ has size at most $e^{k|S|}(k|S|)^{\mathcal{O}(\log k|S|)} k^{k|S|} \log m$. We now describe the derandomized algorithm SRC, which is a result of derandomization of Rand-SRC.

Algorithm SRC. Given an instance (G, S) of Subset Rainbow k-Coloring, the algorithm start by computing an $(m, k|S|, k)$-unified perfect hash family $\mathcal{F}'$. If there exists $c : E(G) \to [k]$, where $c \in \mathcal{F}'$ such that for each $(u, v) \in S$, there is rainbow path between u and v in G' with the edge coloring c then we return c as a solution to Subset Rainbow k-Coloring in (G, S). Otherwise, we return that (G, S) is a *no* instance of Subset Rainbow k-Coloring. We note that for a given graph G with edge coloring c, and vertices u and v, in time $2^k n^{\mathcal{O}(1)}$ time we can check if there is a rainbow path between u and v in G' by using the algorithm given by Corollary 5 in [21]. This completes the description of the algorithm.

▶ **Theorem 24.** *Given an instance (G, k) of* Subset Rainbow k-Coloring*, the algorithm* SRC *either correctly reports that (G, k) is a no instance of* Subset Rainbow k-Coloring *or returns a solution to* Subset Rainbow k-Coloring *in (G, S). Moreover,* SRC *runs in time $2^{\mathcal{O}(|S|)} n^{\mathcal{O}(1)}$, for every fixed k. Here, $n = |V(G)|$.*

Proof. Suppose (G, k) is a *yes* instance of Subset Rainbow k-Coloring, and let $c_F : E(G) \to [k]$ be one of its solution. For $(u, v) \in S$, let P_{uv} be a rainbow path in G'. Furthermore, let $E_R = \cup_{(u,v) \in S} E(P_{uv})$. If $|E_R| < k|S|$, we arbitrarily add edges in G to E_R to make its size exactly $k|S|$. Since $|E_R| \leq k|S|$, there exists $f \in \mathcal{F}$ that splits E_F. Moreover, for each $i \in [k|S|]$, we have $|f^{-1}(i) \cap E_F| \leq 1$. For $i \in [k]$, let $P_i = \{f(e) \mid e \in E_F \text{ and } c_F(e) = i\}$, and $\mathcal{P}' = \{P_i \mid i \in [k]\}$. Notice that $\mathcal{P} = \mathcal{P}' \setminus \{\emptyset\}$ is a partition of $[k|S|]$ into at most k parts. Therefore, the function $f_{\mathcal{P}} \in \mathcal{F}'$. Moreover, $f_{\mathcal{P}}|_{E_F} = c_F|_{E_F}$. The algorithm SRC checks for each $c \in \mathcal{F}'$ whether c is a solution to Subset Rainbow k-Coloring in (G, S). In particular, it checks if $f_{\mathcal{P}}$ is a solution to Subset Rainbow k-Coloring in (G, S). The correctness of this checking is given by Corollary 5 of [21]. Therefore, SRC correctly concludes that (G, S) is a *yes* instance of Subset Rainbow k-Coloring, and outputs a correct solution.

Given an instance (G, k) of Subset Rainbow k-Coloring, whenever it returns a solution then indeed (G, k) is a *yes* instance of Subset Rainbow k-Coloring. This is implied from Corollary 5 of [21].

Next, we move to the runtime analysis. The algorithm starts by computing an $(m, k|S|, k)$-unified perfect hash family $\mathcal{F}'$ of size $e^{k|S|}(k|S|)^{\mathcal{O}(\log k|S|)}k^{k|S|}\log m$ in time $e^{k|S|}(k|S|)^{\mathcal{O}(\log k|S|)}$ $k^{k|S|}$ $m\log m$. Then, for each $c \in \mathcal{F}'$ it checks if for all $(u, v) \in S$, there is a rainbow path between then in G with edge coloring c in time $2^k n^{\mathcal{O}(1)}$. If it finds such a c then returns it as a solution. Otherwise, correctly reports *no*. Therefore, the running time of the algorithm is bounded by $2^{\mathcal{O}(S)}n^{\mathcal{O}(1)}$, for every fixed k. Here, we rely on the fact that $\log |S| \in o(\sqrt{|S|})$. ◀

▶ **Corollary 25.** Steiner Rainbow k-Coloring *admits an algorithm running in time* $2^{\mathcal{O}(|S|^2)}n^{\mathcal{O}(1)}$.

Proof. Follows from Theorem 24. ◀

6 Conclusion

In this paper, we proved that for all $k \geq 3$, Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, unless ETH fails. This (partially) resolves the conjecture of Kowalik et al. [22], which states that for every $k \geq 2$, Rainbow k-Coloring does not admit an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$. It would be an interesting direction to study whether or not Rainbow k-Coloring admits an algorithm running in time $2^{o(|E(G)|)}n^{\mathcal{O}(1)}$, for $k = 2$. We also studied the problem Steiner Rainbow k-Coloring, and proved that for every $k \geq 3$ the problem does not admit an algorithm running in time $2^{o(|S|^2)}n^{\mathcal{O}(1)}$, unless ETH fails. We complemented this by designing an algorithm for Subset Rainbow k-Coloring running in time $2^{\mathcal{O}(|S|)}n^{\mathcal{O}(1)}$, which implies an algorithm running in time $2^{\mathcal{O}(|S|^2)}n^{\mathcal{O}(1)}$ for Steiner Rainbow k-Coloring. It would be interesting to study whether or not Steiner Rainbow k-Coloring admits an algorithm running in time $2^{o(|S|^2)}n^{\mathcal{O}(1)}$, for $k = 2$. Kowalik et al. [22] also conjectured that for every $k \geq 2$, Rainbow k-Coloring does not admit an algorithm running in time $2^{o(n^2)}n^{\mathcal{O}(1)}$, which is another interesting direction of research.

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